पाठ - 7 : समाकलन

महत्वपूर्ण सूत्र

(1) $\int x^n = \frac{x^{n+1}}{n+1} + C.$

(2) $\int dx = x + C.$

(3) $\int \cos x \, dx = \sin x + C.$

(4) $\int \sin x \, dx = -\cos x + C.$

(5) $\int \sec^2 x \, dx = \tan x + C.$

(6) $\int \text{cosec } x \cdot \cot x \, dx = -\text{cosec } x + C.$

(7) $\int \sec x \cdot \tan x \, dx = \sec x + C.$

(8) $\int \text{cosec}^2 x \, dx = -\cot x + C.$

(9) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C.$

(10) $-\int \frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C.$

(11) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C.$

(12) $-\int \frac{1}{1+x^2} dx = \cot^{-1} x + C.$

(13) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C.$

(14) $-\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1}x + C.$

(15) $\int e^x\,dx = e^x + C.$

(16) $\int \frac{1}{x}\,dx = \log x + C.$

(17) $\int \tan x\,dx = -\log|\cos x| + C.$

(18) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C.$

(19) $\int \tan x\,dx = \log|\sec x| + C.$

(20) $\int \cot x\,dx = \log|\sin x| + C.$

(21) $\int \sec x\,dx = \log|\sec x + \tan x| + C.$

(22) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C.$

(23) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$

(24) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C.$

(25) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x + \sqrt{x^2-a^2}| + C.$

(26) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(27) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C$

(28) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

(29) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

(30) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(31) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

(32) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(33) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(34) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(35) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

प्रश्नावली 7.1

निम्नलिखित फलनों के प्रतिअवकलज (समाकलन) निरीक्षण विधि द्वारा ज्ञात कीजिए।

1. $\sin 2x$

उत्तर- $= \int \sin 2x \, dx$

$= \frac{-\cos 2x}{2} + c$

$= \frac{d}{dx}\left(\frac{-\cos 2x}{2} + c\right)$

$= -\left(\frac{-\sin 2x}{2}\right) 2 + 0$

$= \sin 2x$.

2. $\cos 3x$

उत्तर- $= \int \cos 3x \, dx$

$= \frac{\sin 3x}{3} + c$

$= \frac{d}{dx}\left(\frac{\sin 3x}{3} + c\right)$

$= \frac{\cos 3x}{3} \cdot 3 + 0$

$= \cos 3x$

3. e^{2x}

उत्तर- $= \int e^{2x} \cdot dx$

$= \frac{e^{2x}}{2} + C$

$= \frac{d}{dx}\left(\frac{e^{2x}}{2} + C\right)$

$= \frac{e^{2x}}{2} \cdot 2 + 0$

$= e^{2x}$

4. $(ax+b)^2$

उत्तर- $= \int (ax+b)^2 \cdot dx$

$= \frac{(ax+b)^3}{3a}$

$= \frac{d}{dx} \frac{(ax+b)^3}{3a}$

$= \frac{3(ax+b)^2}{3a} \cdot a$

$= (ax+b)^{-}$

5. $\sin 2x - 4e^{3x}$

उत्तर- $= \int (\sin 2x - 4e^{3x}) \cdot dx$

$= \int \sin 2x \cdot dx - \int 4e^{3x} \cdot dx$

$= \frac{-\cos 2x}{2} - \frac{4e^{3x}}{3} + C$

$= \frac{d}{dx}\left(\frac{-\cos 2x}{2} - \frac{4e^{3x}}{3} + C\right)$

$= \frac{-(-\sin 2x)}{2} \cdot 2 - \frac{4e^{3x}}{3} \cdot 3$

$= \sin 2x - 4e^{3x}$

निम्न समाकलनों को ज्ञात कीजिए।

$\int (4e^{3x} + 1)\, dx$

$= \int (4e^{3x} + 1)\, dx$

$= \int 4e^{3x}\, dx + \int 1\, dx$

$= \frac{4e^{3x}}{3} + x + C$

7. $\int x^2\left(1-\frac{1}{x^2}\right)dx$

उत्तर- $=\int\left(x^2-\frac{x^2}{x^2}\right)dx$

$=\int(x^2-1)\,dx$

$=\int x^2\,dx-\int 1\,dx$

$=\frac{x^3}{3}-x+C.$

8. $\int(ax^2+bx+c)\,dx$

उत्तर- $=\int ax^2\,dx+\int bx\,dx+\int c\,dx$

$=a\frac{x^3}{3}+b\frac{x^2}{2}+cx+C$

$=\frac{ax^3}{3}+\frac{bx^2}{2}+cx+C.$

9. $\int(2x^2+e^x)\,dx$

उत्तर- $=\int 2x^2\,dx+\int e^x\,dx$

$=2\frac{x^3}{3}+e^x+C.$

10. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$

उत्तर- $(a-b)^2 = a^2 + b^2 - 2ab$

$$= \int (\sqrt{x})^2 - \left(\frac{1}{\sqrt{x}}\right)^2 - \left(2\sqrt{x} \times \frac{1}{\sqrt{x}}\right) dx$$

$$= \int \left(x + \frac{1}{x} - 2\right) dx$$

$$= \int x\, dx + \int \frac{1}{x}\, dx - \int 2\, dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C.$$

11. $$\int \frac{x^3 + 5x^2 - 4}{x^2}\, dx$$

उत्तर- $$= \int \frac{x^3}{x^2}\, dx + \int \frac{5x^2}{x^2}\, dx - \int \frac{4}{x^2}\, dx$$

$$= \int x\, dx + \int 5\, dx - 4\int x^{-2}\, dx$$

$$= \frac{x^2}{2} + 5x - \frac{4x^{-1}}{-1} + C$$

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C.$$

12. $$\int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$$

उत्तर- $= \int \frac{x^3}{x^{1/2}} dx + \int \frac{3x}{x^{1/2}} dx + \int \frac{4}{x^{1/2}} dx$

$= \int x^{5/2} dx + 3\int x^{1/2} dx + 4\int x^{-1/2} dx$

$= \frac{x^{7/2}}{7/2} + 3\frac{x^{3/2}}{3/2} + 4\frac{x^{1/2}}{1/2} + C$

$= \frac{2}{7} x^{7/2} + 2x^{3/2} + 8\sqrt{x} + C.$

13. $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

उत्तर- $= \int \frac{x^2(x-1) + 1(x-1)}{(x-1)} dx$

$= \int \frac{(x-1)(x^2+1)}{(x-1)} dx$

$= \int x^2 dx + \int 1 dx$

$= \frac{x^3}{3} + x + C.$

14. $\int (1-x)\sqrt{x}\, dx$

उत्तर- $= \int (1-x) x^{1/2} dx$

$= \int x^{1/2} dx - \int x^{3/2} dx$

$$= \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} + C$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C.$$

15. $\int \sqrt{x}\,(3x^2 + 2x + 3)\,dx$

उत्तर - $= \int x^{1/2}\,(3x^2 + 2x + 3)\,dx$

$$= \int 3x^{5/2}\,dx + \int 2x^{3/2}\,dx + \int 3x^{1/2}\,dx$$

$$= 3\frac{x^{7/2}}{7/2} + 2\frac{x^{5/2}}{5/2} + 3\frac{x^{3/2}}{3/2} + C$$

$$= \frac{6}{7}x^{7/2} + \frac{4}{5}x^{5/2} + 2x^{3/2} + C.$$

16. $\int (2x - 3\cos x + e^x)\,dx$

उत्तर - $= \int 2x\,dx - 3\int \cos x\,dx + \int e^x\,dx$

$$= 2\frac{x^2}{2} - 3\sin x + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

उत्तर - $= \int 2x^2\,dx - 3\int \sin x\,dx + 5\int x^{1/2}\,dx$

$= 2\frac{x^3}{3} + 3\cos x + \frac{5x^{3/2}}{3/2} + c$

$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{3/2} + c.$

18. $\int \sec x (\sec x + \tan x)\, dx$

उत्तर- $= \int (\sec^2 x + \sec x \tan x)\, dx$

$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$

$= \tan x + \sec x + c.$

19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx$

उत्तर- $= \int \frac{1}{\cos^2 x} \cdot \frac{\sin^2 x}{1}\, dx$

$= \int \tan^2 x\, dx$

$= \int (\sec^2 x - 1)\, dx$

$= \int \sec^2 x\, dx - \int 1\, dx$

$= \tan x - x + c$

20. $\int \frac{2 - 3\sin x}{\cos^2 x}\, dx$

उत्तर- $= \int \frac{2}{\cos^2 x}\, dx - 3\int \frac{\sin x}{\cos^2 x}\, dx$

$= \int 2\sec^2 x\, dx - 3\int \frac{\sin x}{\cos x \cdot \cos x}\, dx$

$= 2\int \sec^2 x\, dx - 3\int \tan x \cdot \sec x \cdot dx$

$= 2\tan x - 3\sec x + C.$

प्रश्न - 21 एवं 22 में सही उत्तर का चयन कीजिए।

21. $\left[\sqrt{x} + \frac{1}{\sqrt{x}}\right]$ का प्रतिअवकलज है :-

(A) $\frac{1}{3}x^{1/3} + 2x^{1/2} + C.$

(B) $\frac{2}{3}x^{2/3} + \frac{1}{2}x^{2} + C.$

(C) $\frac{2}{3}x^{3/2} + 2x^{1/2} + C.$

(D) $\frac{3}{2}x^{3/2} + \frac{1}{2}x^{1/2} + C.$

उत्तर- $= \int x^{1/2}\, dx + \int x^{-1/2}\, dx$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 2\sqrt{x} + C.$$

22. यदि $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ जिसमें $f(2) = 0$ तो $f(x)$ है :-

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$

(B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$

(D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

उत्तर- $f(x) = \int 4x^3 dx - 3\int x^{-4} dx$

$$f(x) = \frac{4x^4}{4} - \frac{3x^{-3}}{(-3)}$$

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$$f(x) = x^4 + \frac{1}{x^3} + C$$

$$f(2) = (2)^4 + \frac{1}{(2)^3} + C$$

$$0 = \frac{16}{1} + \frac{1}{8} + C$$

$$0 = \frac{129}{8} + C$$

$$C = -\frac{129}{8}$$

$$\boxed{f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}}$$

प्रश्नावली 7.2

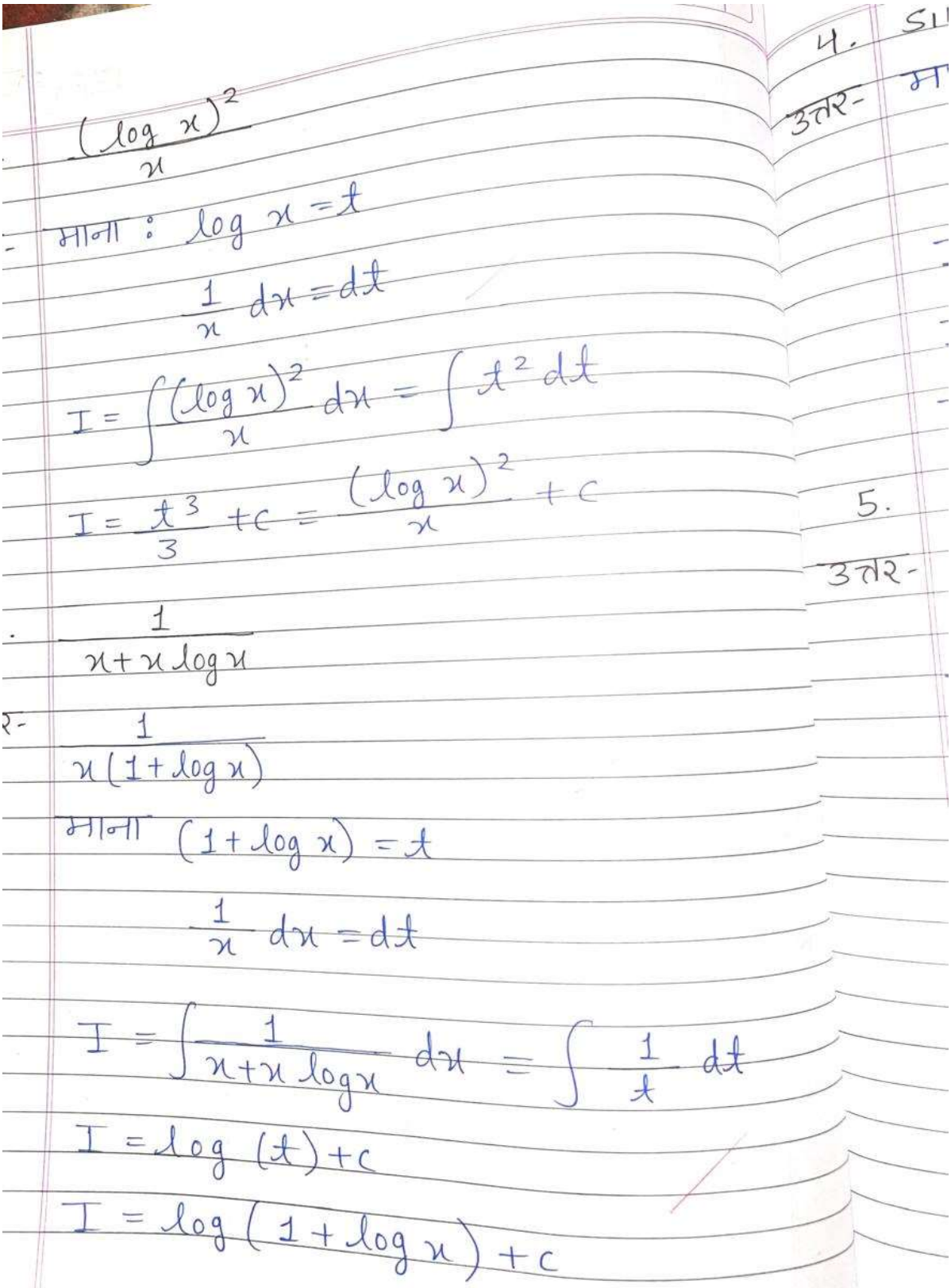
1. $\frac{2x}{1+x^2}$

उत्तर- माना $1+x^2 = t$

$$2x \cdot dx = dt$$

$$I = \int \frac{2x}{1+x^2}\, dx = \int \frac{1}{t}\, dt$$

$$I = \log(t) + C = \log|1+x^2| + C$$



$$\frac{(\log x)^2}{x}$$

माना : $\log x = t$

$$\frac{1}{x}\,dx = dt$$

$$I = \int \frac{(\log x)^2}{x}\,dx = \int t^2\,dt$$

$$I = \frac{t^3}{3} + C = \frac{(\log x)^2}{x} + C$$

$$\frac{1}{x + x\log x}$$

$$\frac{1}{x(1+\log x)}$$

माना $(1+\log x) = t$

$$\frac{1}{x}\,dx = dt$$

$$I = \int \frac{1}{x + x\log x}\,dx = \int \frac{1}{t}\,dt$$

$$I = \log(t) + c$$

$$I = \log(1+\log x) + c$$

4. $\sin x \cdot \sin(\cos x)$

हल- माना $\cos x = t$

$-\sin x \, dx = dt$

$I = \int \sin x \cdot \sin(\cos x) \, dx = -\int \sin t \, dt$

$I = -[-\cos t] + c = \cos t + c$

$I = \cos(\cos x) + c.$

5. $\sin(ax+b)\cos(ax+b)$

हल- $\frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$

माना: $2(ax+b) = t$

$2a \, dx = dt$

$I = \int \frac{\sin 2(ax+b)}{2} \, dx$

$I = \frac{1}{2} \int \frac{\sin t \, dt}{2a}$

$I = \frac{1}{4a}[-\cos t] + c$

$I = -\frac{1}{4a}\cos 2(ax+b) + c$

6. $\sqrt{ax+b}$

उत्तर- माना $ax+b=t$

$a\,dx=dt$

$dx=\frac{1}{a}dt$

$$I=\int (ax+b)^{\frac{1}{2}}\,dx=\frac{1}{a}\int t^{1/2}\,dt$$

$$I=\frac{1}{a}\left(\frac{t^{\frac{1}{2}}}{3/2}\right)+C=\frac{2}{3a}(ax+b)^{3/2}+C.$$

7. $x\sqrt{x+2}$

उत्तर- माना $x+2=t$

$dx=dt$

$x=t-2$.

$$I=\int x\sqrt{x+2}\,dx=\int (t-2)\sqrt{t}\,dt$$

$$I=\int (t^{3/2}-2t^{1/2})\,dt$$

$$I=\int t^{3/2}\,dt-2\int t^{1/2}\,dt$$

$$I=\frac{t^{5/2}}{5/2}-2\left(\frac{t^{3/2}}{3/2}\right)+C$$

$$I=\frac{2}{5}t^{5/2}-\frac{4}{3}t^{3/2}+C$$

$$I = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c.$$

8. $x\sqrt{1+2x^2}$

उत्तर- माना $1+2x^2 = t$

$4x\,dx = dt$

$$I = \int x\sqrt{1+2x^2}\,dx$$

$$I = \frac{1}{4}\int \sqrt{t}\,dt$$

$$I = \frac{1}{4}\int t^{1/2}\,dt$$

$$I = \frac{1}{4}\left(\frac{t^{3/2}}{3/2}\right) + c = \frac{1}{6}(1+2x^2)^{3/2} + c.$$

9. $(4x+2)\sqrt{x^2+x+1}$

उत्तर- माना $x^2+x+1 = t$

$(2x+1)\,dx = dt$

$$I = \int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$I = \int 2\sqrt{t}\,dt = 2\int \sqrt{t}\,dt$$

$$I = 2\left(\frac{t^{3/2}}{3/2}\right) + c$$

$$I = \frac{4}{3}(x^2+x+1)^{3/2} + C.$$

10. $\frac{1}{x-\sqrt{x}}$

उत्तर- $\frac{1}{\sqrt{x}(\sqrt{x}-1)}$

माना $(\sqrt{x}-1) = t$

$$\frac{1}{2\sqrt{x}}\,dx = dt$$

$$I = \int \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx = \int \frac{2}{t}\,dt \times$$

$$I = 2\log|t| + C = 2\log|\sqrt{x}-1| + C.$$

11. $\frac{x}{\sqrt{x+4}}$, $x > 0$

उत्तर- माना $x+4 = t$, $dx = dt$

$$I = \int \frac{x}{\sqrt{x+4}}\,dx = \int \frac{t-4}{\sqrt{t}}\,dt$$

$$I = \frac{t^{3/2}}{3/2} - 4\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$I = \frac{2}{3}t^{3/2} - 8t^{1/2} + C$$

$$I = \frac{2}{3} t \cdot t^{1/2} - 8t^{1/2} + c$$

$$I = \frac{2}{3} t^{1/2} (t - 12) + c$$

$$I = \frac{2}{3} (x+4)^{1/2} (x+4-12) + c$$

$$I = \frac{2}{3} \sqrt{(x+4)}\,(x-8) + c.$$

12. $(x^3-1)^{1/3} x^5$

उत्तर- माना $x^3 - 1 = t$

$3x^2 dx = dt$

$$I = \int (x^3-1)^{1/3} x^5 \, dx = \int (x^3-1)^{1/3} x^3 . x^2 dx$$

$$I = \int t^{1/3} (t+1) \frac{dt}{3} = \frac{1}{3} \int (t^{4/3} + t^{1/3}) dt$$

$$I = \frac{1}{3} \left[\frac{t^{7/3}}{7/3} + \frac{t^{4/3}}{4/3} \right] + c$$

$$I = \frac{1}{3} \left[\frac{3}{7} t^{7/3} + \frac{3}{4} t^{4/3} \right] + c$$

$$I = \frac{1}{7} (x^3-1)^{7/3} + \frac{1}{4} (x^3-1)^{4/3} + c$$

13. $\frac{x^3}{(2+3x^3)^3}$

उत्तर- माना $2+3x^3 = t$

$9x^2 dx = dt$

$$I = \int \frac{x^3}{(2+3x^3)^3} dx = \frac{1}{9} \int \frac{dt}{t^3}$$

$$I = \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + c = \frac{-1}{18}\left(\frac{1}{t^2}\right) + c$$

$$I = \frac{-1}{18(2+3x^3)^2} + c.$$

14. $\frac{1}{x(\log x)^m}$, $x > 0$

उत्तर- माना $\log x = t$

$$\frac{1}{x} dx = dt$$

$$I = \int \frac{1}{x(\log x)^m} dx$$

$$I = \int \frac{dt}{(t)^m} = \left(\frac{t^{-m+1}}{1-m}\right) + c$$

$$I = \frac{(\log x)^{1-m}}{(1-m)} + c$$

15. $\frac{x}{9-4x^2}$

उत्तर- माना $9-4x^2 = t$

$-8x = dt$

$$I = \int \frac{x}{9-4x^2}\,dx = \frac{-1}{8}\int \frac{1}{t}\,dt$$

$$I = \frac{-1}{8}\log|t| + c = \frac{-1}{8}\log|9-4x^2| + c.$$

16. e^{2x+3}

उत्तर- माना $2x+3 = t$

$2dx = dt$

$$I = \int e^{2x+3}\,dx = \frac{1}{2}\int e^t\,dt = \frac{1}{2}(e^t) + c$$

$$I = \frac{1}{2}e^{(2x+3)} + c.$$

17. $\frac{x}{e^{x^2}}$

उत्तर- माना $x^2 = t$

$2x\,dx = dt$

$$I = \int \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int \frac{1}{e^t}\,dt$$

$$I = \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C = -\frac{1}{2}e^{-x^2} + C$$

$$I = \frac{-1}{2e^{x^2}} + C.$$

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$

उत्तर- माना $\tan^{-1}x = t$

$$\frac{1}{1+x^2}dx = dt$$

$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt$$

$$I = e^t + C = e^{\tan^{-1}x} + C$$

19. $\dfrac{e^{2x}-1}{e^{2x}+1}$

उत्तर- अंश और हर को e^x से भाग देने पर

$$\frac{\frac{(e^{2x}-1)}{e^x}}{\frac{(e^{2x}+1)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

माना $e^x + e^{-x} = t$

$(e^x - e^{-x})\,dx = dt$

$$I = \int \frac{e^{2x}-1}{e^{2x}+1}\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$$I = \int \frac{dt}{t} = \log|t| + c = \log|e^x + e^{-x}| + c.$$

20. $\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

उत्तर- माना $e^{2x} + e^{-2x} = t$

$(2e^{2x} - 2e^{-2x})\,dx = dt$

$2(e^{2x} - e^{-2x})\,dx = dt$

$$I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \int \frac{dt}{2t}$$

$$I = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\log|t| + c$$

$$I = \frac{1}{2}\log|e^{2x} + e^{-2x}| + c.$$

21. $\tan^2(2x-3)$

उत्तर- $\tan^2(2x-3) = \sec^2(2x-3) - 1$

माना $2x - 3 = t$, $2\,dx = dt$

$$I = \int \tan^2(2x-3)\,dx$$

$$I = \int [\sec^2(2x-3) - 1]\,dx$$

$$I = \frac{1}{2}\int (\sec^2 t)\,dt - \int 1\,dx$$

$$I = \frac{1}{2}\tan t - x + c$$

$$I = \frac{1}{2}\tan(2x-3) - x + c.$$

22. $\sec^2(7-4x)$

उत्तर- माना $7-4x = t$

$$-4\,dx = dt$$

$$I = \int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$$

$$I = \frac{-1}{4}(\tan t) + c$$

$$I = \frac{-1}{4}\tan(7-4x) + c.$$

23. $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

उत्तर- माना $\sin^{-1} x = t$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$I = \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int t\,dt = \frac{t^2}{2} + c$$

$$I = \frac{(\sin^{-1}x)^2}{2} + c$$

24. $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

उत्तर- $\dfrac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$

माना $(3\cos x + 2\sin x) = t$

$$(-3\sin x + 2\cos x)\,dx = dt$$

$$I = \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\,dx = \int \frac{dt}{2t}$$

$$I = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\log|t| + c$$

$$I = \frac{1}{2}\log|3\cos x + 2\sin x| + c.$$

25. $\dfrac{1}{\cos^2 x\,(1-\tan x)^2}$

उत्तर- $\frac{\sec^2 x}{(1-\tan x)^2}$

माना $(1-\tan x) = t$

$-\sec^2 x\, dx = dt$

$I = \int \frac{\sec^2 x}{(1-\tan x)^2}\, dx = \int \frac{-dt}{t^2}$

$I = -\int t^{-2}\, dt = \frac{1}{t} + C$

$I = \frac{1}{1-\tan x} + C$

26. $\frac{\cos\sqrt{x}}{\sqrt{x}}$

उत्तर- माना $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}}\, dx = dt$

$I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = 2\int \cos t\, dt$

$I = 2\sin t + c$

$I = 2\sin\sqrt{x} + c$

27. $\sqrt{\sin 2x}\cos 2x$

उत्तर- माना $\sin 2x = t$

$\cos 2x\, dx = dt$

$I = \int \sqrt{\sin 2x}$

$I = \frac{1}{2}\int \sqrt{t}\, dt = \frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + c$

$I = \frac{1}{3} t^{3/2} + c = \frac{1}{3}(\sin 2x)^{3/2} + c.$

28. $\frac{\cos x}{\sqrt{1+\sin x}}$

उत्तर- माना $1 + \sin x = t$

$\cos x\, dx = dt$

$I = \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$

$I = \frac{t^{1/2}}{1/2} + c = 2\sqrt{t} + c$

$I = 2\sqrt{1+\sin x} + c.$

29. $\cot x \log \sin x$

उत्तर- माना $\log \sin x = t$

$$\frac{1}{\sin x} \cdot \cos x \, dx = dt$$

$$\cot x \, dx = dt$$

$$I = \int \cot x \log \sin x \, dx$$

$$I = \int t \, dt = \frac{t^2}{2} + c = \frac{(\log \sin x)^2}{2} + c.$$

30. $\frac{\sin x}{1 + \cos x}$

उत्तर- $1 + \cos x = t$

$$-\sin x \, dx = dt$$

$$I = \int \frac{\sin x}{1 + \cos x} dx = \int -\frac{dt}{t}$$

$$I = -\log |t| + c$$

$$I = -\log |1 + \cos x| + c.$$

31. $\frac{\sin x}{(1 + \cos x)^2}$

उत्तर- माना $1 + \cos x = t$

$$-\sin x \, dx = dt$$

$$I = \int \frac{\sin x}{(1+\cos x)^2}\,dx = \int \frac{-dt}{t^2}$$

$$I = -\int t^{-2}\,dt = \frac{1}{t} + c$$

$$I = \frac{1}{1+\cos x} + c.$$

32. $\dfrac{1}{1+\cot x}$

हल- $$I = \int \frac{1}{1+\cot x}\,dx = \int \frac{1}{1+\frac{\cos x}{\sin x}}\,dx$$

$$I = \int \frac{\sin x}{\sin x+\cos x}\,dx = \frac{1}{2}\int \frac{2\sin x\,dx}{\sin x+\cos x}$$

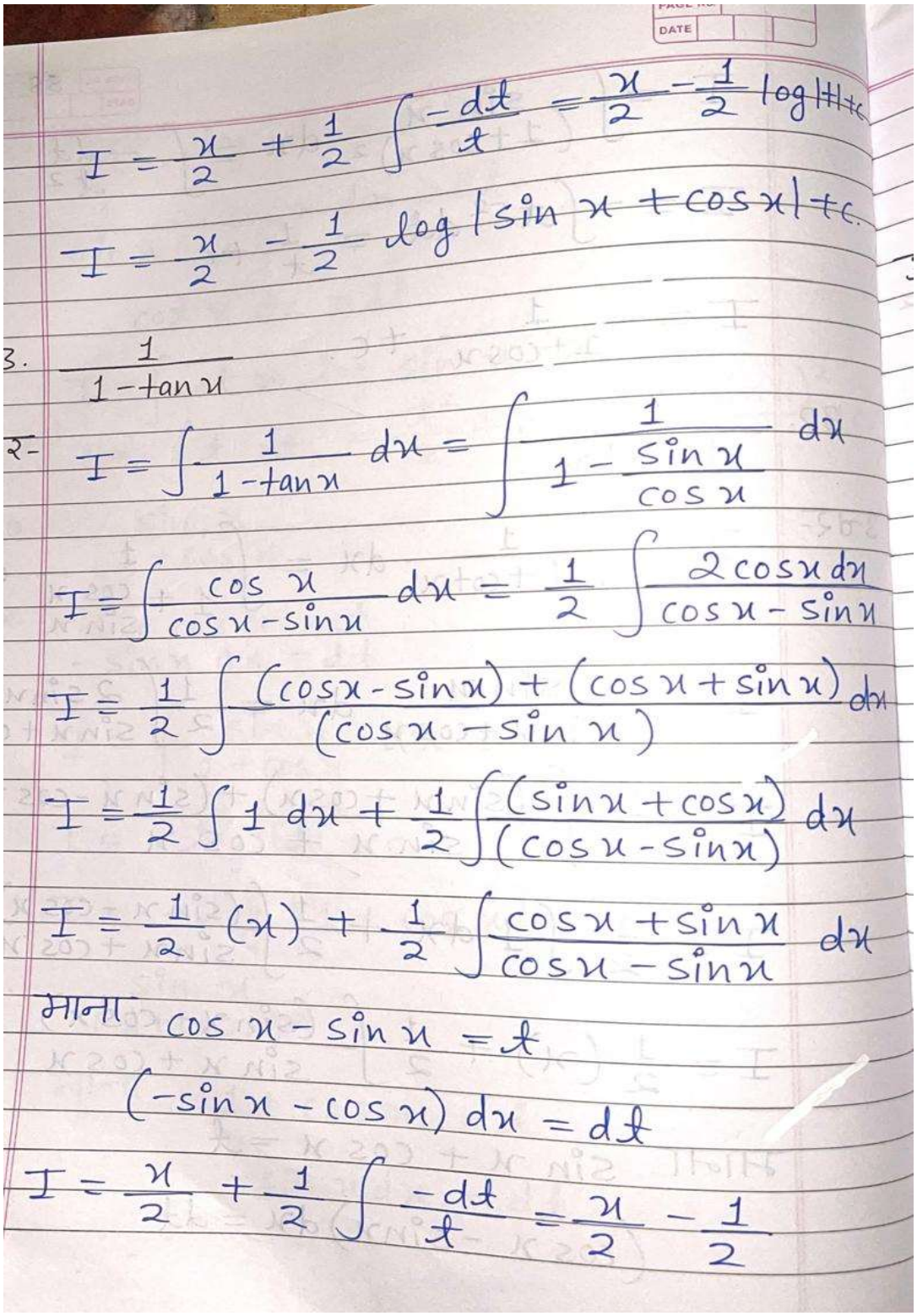
$$I = \frac{1}{2}\int \frac{(\sin x+\cos x)+(\sin x-\cos x)}{\sin x+\cos x}\,dx$$

$$I = \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{(\sin x-\cos x)\,dx}{\sin x+\cos x}$$

$$I = \frac{1}{2}(x) + \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x+\cos x}\,dx$$

माना $\sin x + \cos x = t$

$(\cos x - \sin x)\,dx = dt$



$$I = \frac{x}{2} + \frac{1}{2}\int \frac{-dt}{t} = \frac{x}{2} - \frac{1}{2}\log|t| + c$$

$$I = \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + c.$$

3. $\frac{1}{1-\tan x}$

हल- $I = \int \frac{1}{1-\tan x}\,dx = \int \frac{1}{1-\frac{\sin x}{\cos x}}\,dx$

$$I = \int \frac{\cos x}{\cos x - \sin x}\,dx = \frac{1}{2}\int \frac{2\cos x\,dx}{\cos x - \sin x}$$

$$I = \frac{1}{2}\int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)}\,dx$$

$$I = \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{(\sin x + \cos x)}{(\cos x - \sin x)}\,dx$$

$$I = \frac{1}{2}(x) + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}\,dx$$

माना $\cos x - \sin x = t$

$$(-\sin x - \cos x)\,dx = dt$$

$$I = \frac{x}{2} + \frac{1}{2}\int \frac{-dt}{t} = \frac{x}{2} - \frac{1}{2}$$

$\log|\cos x - \sin x| + c.$

34. $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

उत्तर- $I = \int \dfrac{\sqrt{\tan x}}{\sin x \cos x}\, dx$

$I = \int \dfrac{\sqrt{\tan x}}{\sin \cos x} \times \dfrac{\cos x}{\cos x}\, dx$

$I = \int \dfrac{\sqrt{\tan x}}{\tan x \cos^2 x}\, dx = \int \dfrac{\sec^2 x}{\sqrt{\tan x}}\, dx$

माना $\tan x = t$

$\sec^2 x\, dx = dt$

$I = \int \dfrac{dt}{\sqrt{t}} = 2\sqrt{t} + c = 2\sqrt{\tan x} + c.$

35. $\dfrac{(1+\log x)^2}{x}$

उत्तर- माना $(1+\log x) = t$

$\dfrac{1}{x}\, dx = dt$

$I = \int \dfrac{(1+\log x)^2}{x}\, dx = \int t^2\, dt$

$$I = \frac{t^3}{3} + c = \frac{(1+\log x)^3}{3} + c.$$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$

उत्तर- $\dfrac{(x+1)}{x}(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$

माना $(x+\log x) = t$

$$\left(1+\frac{1}{x}\right)dx = dt$$

$$I = \int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx$$

$$I = \int t^2 dt$$

$$I = \frac{t^3}{3} + c = \frac{(x+\log x)^3}{3} + c.$$

37. $\dfrac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$

उत्तर- माना $x^4 = t$, $4x^3 dx = dt$

$$I = \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} dx$$

$$I = \frac{1}{4}\int \frac{\sin(\tan^{-1} t)}{1+t^2} \quad \text{— समी. 1}$$

माना $\tan^{-1} t = u$

$$\frac{1}{1+t^2}\,dt = du$$

समी. 1 से

$$I = \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\,dx$$

$$I = \frac{1}{4}\int \sin u\,du = \frac{1}{4}(-\cos u) + C$$

$$I = \frac{-1}{4}\cos(\tan^{-1} t) + C$$

$$I = \frac{-1}{4}\cos(\tan^{-1} x^4) + C.$$

38. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx$ बराबर है :

(A) $10^x - x^{10} + C$
(B) $10^x + x^{10} + C$
(C) $(10^x - x^{10})^{-1} + C$
(D) $\log(10^x + x^{10}) + C$

उत्तर- माना $x^{10} + 10^x = t$, $(10x^9 + 10^x \log_e 10)\,dx = dt$

$$I = \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\, dx$$

$$I = \int \frac{dt}{t} = \log t + C$$

$$I = \log (10^x + x^{10}) + C$$

अतः विकल्प (D) सही है।

प्रश्न-39: $\int \frac{dx}{\sin^2 x \cos^2 x}$ बराबर है :-

(A) $\tan x + \cot x + C$
(B) $\tan x - \cot x + C$
(C) $\tan x \cot x + C$
(D) $\tan x \cot 2x + C$

उत्तर- $I = \int \frac{dx}{\sin^2 \cos^2 x}\, dx$

$$I = \int \frac{1}{\sin^2 \cos^2 x}\, dx$$

$$I = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$I = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx + \int \frac{\cos^2 x\, dx}{\sin^2 x \cos^2 x}$$

$$I = \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$I = \tan x - \cot x + c$

अतः विकल्प (B) सही है।

(1) $2\cos^2 x = 1 + \cos 2x$

(2) $\cos^2 x = \frac{1+\cos 2x}{2}$

(3) $2\sin^2 x = 1 - \cos 2x$

(4) $\sin^2 x = \frac{1-\cos 2x}{2}$

(5) $\cos 2x = \cos^2 x - \sin^2 x$

(6) $\sin 3x = 3\sin x - 4\sin^3 x$

(7) $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

(8) $\tan^2 x = \sec^2 x - 1$

(9) $1 - \cos x = 2\sin^2 \frac{x}{2}$

(10) $1 + \cos x = 2\cos^2 \frac{x}{2}$

(11) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(12) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(13) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(14) $\sin A \sin B = \dfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

(15) $\cos A \cos B = \dfrac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

(16) $\sin A \cos B = \dfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

(17) $\cos A \sin B = \dfrac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

(18) $\sin A + \sin B = 2 \sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(19) $\sin A - \sin B = 2 \cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

(20) $\cos A + \cos B = 2 \cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(21) $\cos A - \cos B = -2 \sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

प्रश्नावली 7.3

1. $\sin^2(2x+5)$

उत्तर- $\sin^2(2x+5)$

$= \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$

$I = \int \sin^2(2x+5)\, dx$

$I = \int \frac{1-\cos(4x+10)}{2}\, dx$

$I = \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos(4x+10)\, dx$

$I = \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$

$I = \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C.$

2. $\sin 3x \cdot \cos 4x$

उत्तर- $\left\{\sin A \cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]\right\}$

$I = \int \sin 3x \cdot \cos 4x\, dx$

$$I = \frac{1}{2}\int (\sin(3x+4x) + \sin(3x-4x))\,dx$$

$$I = \frac{1}{2}\{\sin 7x + \sin(-x)\}\,dx$$

$$I = \frac{1}{2}(\sin 7x - \sin x)\,dx$$

$$I = \frac{1}{2}\int \sin 7x\,dx - \frac{1}{2}\int \sin x\,dx$$

$$I = \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + c$$

$$I = -\frac{\cos 7x}{14} + \frac{\cos x}{2} + c$$

3. $\cos 2x \cos 4x \cos 6x$

उत्तर-

$$\left[\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}\right.$$

$$I = \int \cos 2x\,(\cos 4x \cos 6x)\,dx$$

$$I = \int \cos 2x \left[\frac{1}{2}(\cos 2x\,(4x+6x) + \cos(4x-6x))\right] dx$$

$$I = \frac{1}{2}\int \{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}\, dx$$

$$I = \frac{1}{2}\int \{\cos 2x \cos 10x + \cos^2 2x\}\, dx$$

$$I = \frac{1}{2}\int \left[\frac{1}{2}\{\cos(2x+10x) + \cos(2x-10x)\} + \left\{\frac{1+\cos 4x}{2}\right\}\right] dx$$

$$I = \frac{1}{4}\int (\cos 12x + \cos 8x + 1 + \cos 4x)\, dx$$

$$I = \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4}\right] + C$$

4. $\sin^3(2x+1)$

उत्तर-

$$I = \int \sin^3(2x+1)\, dx$$

$$I = \int \sin^2(2x+1) \cdot \sin(2x+1)\, dx$$

$$I = \int \{1 - \cos^2(2x+1)\} \sin(2x+1)\, dx$$

माना $\cos(2x+1) = t$

$-2 \sin(2x+1)\,dx = dt$

$\sin(2x+1)\,dx = \frac{-dt}{2}$

$I = \frac{-1}{2}\int (1-t^2)\,dt = \frac{-1}{2}\left\{t - \frac{t^3}{3}\right\} + c$

$I = \frac{-1}{2}\left\{\cos(2x+1) - \frac{\cos(2x+1)^3}{3}\right\} + c$

$I = \frac{-\cos(2x+1)}{2} + \frac{\cos(2x+1)^3}{3} + c$

5. $\sin^3 x \cos^3 x$

उत्तर- $I = \int \sin^3 x \cos^3 x\,dx$

$I = \int \cos^3 x \cdot \sin^2 x \cdot \sin x\,x\,dx$

$I = \int \cos^3 x (1-\cos^2 x) \cdot \sin x\,dx$

माना $\cos x = t$

$-\sin x \cdot dx = dt$

$I = -\int t^3 (1-t^2)\,dt$

$I = -\int (t^3 - t^5)\,dt$

$$I = -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$

$$I = -\left(\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right) + C$$

$$I = \frac{\cos^4 x}{4} - \frac{\cos^6 x}{6} + C$$

6. $\sin x \sin 2x \sin 3x$

HINT- $\left\{\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]\right\}$

$$I = \int \sin x \sin 2x \sin 3x \, dx$$

$$I = \int \left[\sin \frac{1}{2}\left\{\cos(2x-3x) - \cos(2x+3x)\right\}\right] dx$$

$$I = \frac{1}{2}\int \left\{\sin x \cos(-x) - \sin x \cos 5x\right\} dx$$

$$I = \frac{1}{2}\int (\sin x \cos x - \sin x \cos 5x)\, dx$$

$$I = \frac{1}{2}\int \frac{\sin 2x}{2}\, dx - \frac{1}{2}\int \sin x \cos 5x \, dx$$

$$I = \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int\left\{\frac{1}{2}\sin(x+5x) + \sin(x-5x)\right\}dx$$

$$I = \frac{-\cos 2x}{8} - \frac{1}{4}\int\{\sin 6x + \sin(-4x)\}dx$$

$$I = \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + c$$

$$I = \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + c$$

$$I = \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + c$$

7. $\sin 4x \sin 8x$

उत्तर- $\left\{\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}\right\}$

$$I = \int \sin 4x \sin 8x \, dx$$

$$I = \int \frac{1}{2}\{\cos(4x-8x) - \cos(4x+8x)\}dx$$

$$I = \frac{1}{2}\int \{\cos(-4x) - \cos 12x\}\, dx$$

$$I = \frac{1}{2}\int (\cos 4x - \cos 12x)\, dx$$

$$I = \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + c$$

8. $\dfrac{1-\cos x}{1+\cos x}$

3 मार्क

$$= \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \tan^2 \frac{x}{2} = \sec^2 \frac{x}{2} - 1$$

$$I = \int \frac{1-\cos x}{1+\cos x}\, dx$$

$$I = \int \left\{\sec^2 \frac{x}{2} - 1\right\} dx$$

$$I = \left[\frac{\tan x/2}{1/2} - 1\right] + c$$

$$I = 2\tan \frac{x}{2} + c$$

9. $\frac{\cos x}{1+\cos x}$

उत्तर- $= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \frac{1}{2}\left[1 - \tan^2 \frac{x}{2}\right]$

$I = \int \frac{\cos x}{1+\cos x}\, dx = \frac{1}{2}\int \left(1 - \tan^2 \frac{x}{2}\right) dx$

$I = \frac{1}{2}\left(1 - \sec^2 \frac{x}{2} + 1\right) dx$

$I = \frac{1}{2}\left(2 - \sec^2 \frac{x}{2}\right) dx$

$I = \frac{1}{2}\left[2x - \frac{\tan x/2}{1/2}\right] + c$

$I = x - \tan \frac{x}{2} + c$

10. $\sin^4 x$.

उत्तर- $= \sin^2 x \sin^2 x$

$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$

$= \frac{1}{4}(1-\cos 2x)^2$

$$= \frac{1}{4}\left[1 + \cos^2 2x - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$I = \int \sin^4 x\, dx = \frac{1}{4}\int\left[\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right] dx = \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\left(\frac{\sin 4x}{4}\right) - 2\,\frac{2\sin 2x}{2}\right] + c$$

$$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - 2\sin 2x\right] + c$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

11. $\cos^4 2x$

12. $= (\cos^2 2x)^2$

$$= \left(\frac{1+\cos 4x}{2}\right)^2$$

$$= \frac{1}{4}\left[1 + \cos^2 4x + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1+\cos 8x}{2}\right) + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{2}\cos 8x + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\cos 8x + 2\cos 4x\right]$$

$$I = \int \cos^4 2x \, dx$$

$$I = \int \left[\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right] dx$$

$$I = \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

12. $\dfrac{\sin^2 x}{1+\cos x}$

उत्तर-

$$= \frac{\left(2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$= 2\sin^2 x/2$

$= 1 - \cos x$

$$I = \int \frac{\sin^2 x}{1+\cos x}\, dx$$

$$I = \int (1-\cos x)\, dx$$

$$I = x - \sin x + c$$

3. $$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$

$$= \frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}}$$

$$\left[\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cdot\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\left[\left(\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right)\right]\right]$$

$$= 2\left[\cos(x)+\cos\alpha\right] = 2\cos x + 2\cos\alpha$$

$$I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos a}\, dx$$

$$I = \int (2\cos x + 2\cos\alpha)\, dx$$

$$I = 2\left[\sin x + x\cos\alpha\right] + C$$

4. $$\frac{\cos x - \sin x}{1+\sin 2x}$$

2. $$= \frac{\cos x - \sin x}{(\sin^2 x + \cos^2 x) + 2\sin x \cos x}$$

$$= \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

$$I = \int \frac{\cos x - \sin x}{1+\sin 2x}\, dx$$

$$I = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}\, dx$$

माना $\sin x + \cos x = t$

$$\cos x - \sin x\, dx = dt$$

$$I = \int \frac{dt}{t^2} = \int t^{-2}\, dt$$

$$I = -t^{-1} + c$$

$$I = \frac{-1}{t} + c$$

$$I = \frac{-1}{\sin x + \cos x} + c$$

15. $\tan^3 2x \cdot \sec 2x$

उत्तर- $= \tan^2 2x \tan 2x \sec 2x$

$= (\sec^2 2x - 1) \tan 2x \sec 2x$

$= \sec^2 2x \tan 2x \sec 2x - \tan 2x \sec 2x$

$$I = \int \tan^3 2x \cdot \sec 2x\, dx$$

$$I = \int \tan^2 2x \tan 2x \sec 2x\, dx - \int \tan 2x \sec 2x\, dx$$

$$I = \int \tan^2 2x \tan 2x \sec 2x \, dx$$

$$- \frac{\sec 2x}{2} + c$$

माना $\sec 2x = t$

$$2 \sec 2x \tan 2x \, dx = dt$$

$$I = \int \tan^3 2x \cdot \sec 2x \, dx$$

$$I = \frac{1}{2} \int t^2 dt \quad - \frac{\sec 2x}{2} + c$$

$$I = \frac{t^3}{6} - \frac{\sec 2x}{2} + c$$

$$I = \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + c$$

16. $\tan^4 x$

उत्तर- $= \tan^2 x \tan^2 x$

$= (\sec^2 x - 1) \tan^2 x$

$= \sec^2 x \cdot \tan^2 x - \tan^2 x$

$= \sec^2 x \cdot \tan^2 x - (\sec^2 x - 1)$

$= \sec^2 x \cdot \tan^2 x - \sec^2 x + 1$

$$I = \int \tan^4 x\, dx = \int \sec^2 x \tan^2 x\, dx - \int \sec^2 x\, dx + \int 1\, dx$$

$$I = \int \sec^2 x \cdot \tan^2 x\, dx - \tan x + x + c \quad -(1)$$

माना $I_1 = \int \sec^2 x \tan^2 x\, dx$

माना $\tan x = t$

$$\sec^2 x\, dx = dt$$

$$I_1 = \int \sec^2 x \tan^2 x\, dx = \int t^2\, dt$$

$$I_1 = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

समी॰ 1 से, $I = \int \tan^4 x\, dx$

$$I = \frac{1}{3}\tan^3 x - \tan x + x + c$$

17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^3 x \cdot \cos^3 x}$

हल-

$$= \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$= \tan x \sec x + \cot x \operatorname{cosec} x$

$$I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cdot \cos^2 x} dx$$

$$I = \int (\tan x \sec x + \cot x \operatorname{cosec} x) dx$$

$$I = \sec x - \operatorname{cosec} x + C$$

18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

उत्तर- $= \dfrac{\cos 2x + (1 - \cos 2x)}{\cos^2 x}$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$$

$$I = \int \sec^2 x \, dx = \tan x + C.$$

19. $\dfrac{1}{\sin x \cos^3 x}$

उत्तर- $= \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$

$$= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$= \tan x \sec^2 x + \frac{\cos^2 x}{\left(\frac{\sin x \cos x}{\cos^2 x}\right)}$$

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

$$I = \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x$$

$$dx + \int \frac{\sec^2 x}{\tan x} dx$$

माना $\tan x = t$

$$\sec^2 x \, dx = dt$$

$$I = \int \frac{1}{\sin x \cos^3 x} dx = \int t \, dt +$$

$$\int \frac{1}{t} dt = \frac{t^2}{2} + \log |t| + c$$

$$= \frac{1}{2} \tan^2 x + \log |\tan x| + c.$$

20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

उत्तर-

$$= \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2 \sin x \cos x}$$

$$= \frac{\cos 2x}{1+\sin 2x}$$

$$I = \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx$$

$$I = \int \frac{\cos 2x}{1+\sin 2x}\, dx$$

माना $(1 + \sin 2x) = t$

$$2\cos 2x\, dx = dt$$

$$I = \int \frac{\cos 2x}{(\cos x+\sin x)^2}\, dx$$

$$I = \frac{1}{2}\int \frac{1}{t}\, dt = \frac{1}{2}\log|t| + C.$$

$$I = \frac{1}{2}\log|1+\sin 2x| + C.$$

$$I = \frac{1}{2}\log|(\cos x+\sin x)^2| + C$$

$$I = \log|\cos x + \sin x| + c$$

21. $\sin^{-1}(\cos x)$

उत्तर- $I = \int \sin^{-1}(\cos x)\, dx$

$$I = \int \sin^{-1}\left(\sin\left\{\frac{\pi}{2} - x\right\}\right) dx$$

$$I = \int \left\{\frac{\pi}{2} - x\right\} dx$$

$$I = \frac{\pi}{2}x - \frac{x^2}{2} + C.$$

22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

उत्तर-

$$= \frac{1}{\sin(x-a)} \left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(x-a)} \left[\frac{\sin\{(x-b)-(x-a)\}}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(x-a)} \left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(x-a)} \left[\tan(x-b) - \tan(x-a)\right]$$

$$I = \int \frac{1}{\cos(x-a)\cos(x-b)} dx$$

$$I = \frac{1}{\sin(x-a)} \int \tan(x-b) - \tan(x-a)\, dx$$

$$I = \frac{1}{\sin(x-a)} \left[-\log|\cos(x-b)| + \log|\cos(x-a)|\right]$$

$$I = \frac{1}{\sin(x-a)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$$

23. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 \cos^2 x}\, dx$ बराबर है :

(A) $\tan x + \cot x + C$
(B) $\tan x + \operatorname{cosec} x + C$
(C) $-\tan x + \cot x + C$
(D) $\tan x + \sec x + C$

उत्तर- $I = \int \frac{\sin^2 x - \cos^2 x}{\sin^2 \cos^2 x}\, dx$

$$I = \int \left(\frac{\sin^2 x}{\sin^2 \cos^2 x} - \frac{\cos^2 x}{\sin^2 \cos^2 x}\right) d$$

$$I = \int (\sec^2 x - \operatorname{cosec}^2 x)\, dx$$

$$I = \tan x + \cot x + C$$

अतः विकल्प (A) सही है।

24. $\int \frac{e^x (1+x)}{\cos^2 (e^x x)}$ बराबर है :

(A) $-\cot (e^x x) + C$
(B) $\tan (e^x x) + C$
(C) $\tan (e^x) + C$
(D) $\cot (e^x) + C$

उत्तर- $I = \int \frac{e^x(1+x)}{\cos^2(e^x x)}\, dx$

माना $e^x x = 1$

$(e^x . x + e^x . 1)\, dx = dt$

$e^x (x+1)\, dx = dt$

$I = \int \frac{e^x(1+x)}{\cos^2(e^x x)}\, dx = \int \frac{dt}{\cos^2 t}$

$I = \int \sec^2 t\, dt$

$I = \tan(e^x . x) + C$

अतः विकल्प (B) सही है।

(1) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

(2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

(3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(4) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log \left| x + \sqrt{x^2-a^2} \right| + c$

(5) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$

(6) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log \left| x + \sqrt{x^2+a^2} \right| + c$

प्रश्नावली 7.4

1. $\frac{3x^2}{x^6+1}$

उत्तर- माना $x^3 = t$

$3x^2 dx = dt$

$I = \int \frac{3x^2}{x^6+1} dx$

$I = \int \frac{dt}{t^2+1}$

$I = \tan^{-1} t + C$

$I = \tan^{-1}(x^3) + C$

2. $\dfrac{1}{\sqrt{1+4x^2}}$

उत्तर- माना $2x = t$

$2\,dx = dt$

$I = \int \dfrac{1}{\sqrt{1+4x^2}}\,dx$

$I = \dfrac{1}{2}\int \dfrac{1}{\sqrt{1+t^2}}\,dt$

$I = \dfrac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C$

$I = \dfrac{1}{2}\left[\log\left|2x+\sqrt{4x^2+1}\right|\right] + C$

3. $\dfrac{1}{\sqrt{(2-x)^2+1}}$

उत्तर- माना $2-x = t$

$-dx = dt$

$I = \int \dfrac{1}{\sqrt{(2-x)^2+1}}\,dx$

$$I = -\int \frac{1}{\sqrt{t^2+1}}\,dt$$

$$I = -\log\left| t + \sqrt{t^2+1} \right| + C$$

$$I = -\log\left| 2-x + \sqrt{(2-x)^2+1} \right| + C$$

$$I = \log\left| \frac{1}{(2-x) + \sqrt{x^2-4x+5}} \right| + C$$

4. $\dfrac{1}{\sqrt{9-25x^2}}$

उत्तर- माना $5x = t$

$5\,dx = dt$

$$I = \int \frac{1}{\sqrt{9-25x^2}}\,dx$$

$$I = \frac{1}{5}\int \frac{1}{\sqrt{9-t^2}}\,dt$$

$$I = \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$

$$I = \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$

$$I = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

5. $\frac{3x}{1+2x^4}$

उत्तर- माना $\sqrt{2}\,x^2 = t$

$2\sqrt{2}\,x\,dx = dt$

$$I = \int \frac{3x}{1+2x^4}\,dx$$

$$I = \frac{2}{2\sqrt{2}} \int \frac{dt}{1+t^2}$$

$$I = \frac{2}{2\sqrt{2}} \left(\tan^{-1} t\right) + C$$

$$I = \frac{2}{2\sqrt{2}} \left[\tan^{-1}\left(\sqrt{2}\,x^2\right)\right] + C$$

6. $\frac{x^2}{1-x^6}$

उत्तर- माना $x^3 = t$

$3t^2\,dx = dt$

$$I = \int \frac{x^2}{1-x^6}\,dx$$

$$I = \frac{1}{3} \int \frac{dt}{1-t^2}$$

$$I = \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

$$I = \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

7. $\dfrac{x-1}{\sqrt{x^2-1}}$

उत्तर- $I = \int \dfrac{x-1}{\sqrt{x^2-1}}\,dx$

$$I = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \quad \ldots (1)$$

$\int \dfrac{x}{\sqrt{x^2-1}}\,dx$ के लिए,

माना $x^2 - 1 = t$

$2x\,dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-1/2}\,dt = \frac{1}{2}\left[2t^{1/2}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

$$I = \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

$$I = \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

7. $\dfrac{x-1}{\sqrt{x^2-1}}$

उत्तर- $I = \int \dfrac{x-1}{\sqrt{x^2-1}}\,dx$

$$I = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \quad \ldots(1)$$

$\int \dfrac{x}{\sqrt{x^2-1}}\,dx$ के लिए,

माना $x^2-1 = t$

$2x\,dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-1/2}\,dt = \frac{1}{2}\left[2t^{1/2}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

समी. 1 से

$$I = \int \frac{x-1}{\sqrt{x^2-1}}\,dx$$

$$I = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx$$

$$I = \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C.$$

8. $\dfrac{x^2}{\sqrt{x^6+a^6}}$

उत्तर- माना $x^3 = t$

$$3x^2\,dx = dt$$

$$I = \int \frac{x^2}{\sqrt{x^6+a^6}}\,dx$$

$$I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+(a^3)^2}}$$

$$I = \frac{1}{3}\log\left|t+\sqrt{t^2+(a^3)^2}\right| + C$$

$$I = \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

9. $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

उत्तर- माना $\tan x = t$

$\sec^2 x \, dx = dt$

$$I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$$

$$I = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$I = \log | t + \sqrt{t^2 + 2^2} | + c$$

$$I = \log | \tan + \sqrt{\tan^2 x + 4} | + c$$

10. $\frac{1}{\sqrt{x^2 + 2x + 4}}$

उत्तर- $$I = \int \frac{1}{\sqrt{x^2 + 2x + 4}} dx$$

$$I = \int \frac{1}{\sqrt{(x+1)^2 + 1^2}} dx$$

माना $x + 1 = t$

$dx = dt$

$$I = \int \frac{1}{\sqrt{x^2 + 2x + 4}} dx$$

$$I = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$I = \log \left| t + \sqrt{t^2+1} \right| + C$$

$$I = \log \left| (x+1) + \sqrt{(x+1)^2+1} \right| + C$$

$$I = \log \left| (x+1) + \sqrt{x^2+2x+4} \right| + C$$

11. $\dfrac{1}{9x^2+6x+5}$

उत्तर- $I = \int \dfrac{1}{9x^2+6x+5}\, dx$

$$I = \int \frac{1}{(3x+1)^2+2^2}$$

माना $(3x+1) = t$

$$3\,dx = dt$$

$$I = \int \frac{1}{(3x+1)^2+2^2}\, dx$$

$$I = \frac{1}{3}\int \frac{1}{t^2+2^2}\, dt$$

$$I = \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] + C$$

$$I = \frac{1}{6}\tan^{-1}\left(\frac{t}{2}\right) + C$$

12. $\frac{1}{\sqrt{7-6x-x^2}}$

उत्तर-

$= 7 - 6x - x^2$

$= 7 - (x^2 + 6x + 9 - 9)$

$= 7 - (x^2 + 6x + 9 - 9)$

$= 16 - (x^2 + 6x + 9)$

$= 16 - (x+3)^2$

$= (4)^2 - (x+3)^2$

$\therefore\ I = \int \frac{1}{\sqrt{7-6x-x^2}}\, dx$

$I = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}}\, dx$

माना $x+3 = t$

$dx = dt$

$I = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}}\, dx$

$I = \int \frac{1}{\sqrt{(4)^2 - (t)^2}}\, dt$

$I = \sin^{-1}\left(\frac{t}{4}\right) + c$

$I = \sin^{-1}\left(\frac{x+3}{4}\right) + c$

13. $\frac{1}{\sqrt{(x-1)(x-2)}}$

उत्तर- $(x-1)(x-2) = x^2-3x+2$

$= x^2-3x+9/4-9/4+2$

$= \left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{4}\right)$

$= \left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$I = \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$

माना $x-\frac{3}{2} = t$

$dx = dt$

$I = \int \frac{1}{\sqrt{(t)^2-\left(\frac{1}{2}\right)^2}}\,dt$

$I = \log\left|t+\sqrt{(t)^2-\left(\frac{1}{2}\right)^2}\right| + c$

$$I = \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+c$$

14. $\dfrac{1}{\sqrt{8+3x-x^2}}$

उत्तर- $= 8+3x-x^2$

$$= 8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$= \frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$I = \int \frac{1}{\sqrt{8+3x-x^2}}\,dx$$

$$I = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

माना $x-\frac{3}{2} = t$

$$dx = dt$$

$$I = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - t^2}}\, dt$$

$$I = \sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right) + C$$

$$I = \sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}}\right) + C$$

$$I = \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C$$

15. $\frac{1}{\sqrt{(x-a)(x-b)}}$

उत्तर- $(x-a)(x-b) = x^2 - (x-b)x + ab$

$$= x^2 - (x-b)x + \frac{(a+b)^2}{4} - \frac{(a+b)^2}{4} + ab$$

$$= \left[x - \left(\frac{a+b}{2}\right)\right]^2 - \frac{(a+b)^2}{4}$$

$$I = \int \frac{1}{\sqrt{(x-a)(x-b)}}\, dx$$

$$I = \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2 - \left(\frac{a+b}{4}\right)^2}}\, dx$$

माना $x - \left(\frac{a+b}{2}\right) = t$

$$dx = dt$$

$$I = \int \frac{1}{\sqrt{t^2 - \left(\frac{a+b}{2}\right)^2}}\, dt$$

$$I = \log \left| t + \sqrt{t^2 - \left(\frac{a+b}{2}\right)^2} \right| + C$$

$$I = \log \left| \left\{x - \left(\frac{a+b}{2}\right)\right\} + \sqrt{(x-a)(x-b)} \right| + C$$

प्रश्न-16 : $\frac{4x+1}{\sqrt{2x^2+x-3}}$

उत्तर- माना $4x+1 = A\frac{d}{dx}(2x^2+x-3)+B$

$$4x+1 = A(4x+1)+B$$

$$4x+1 = 4Ax + A + B$$

दोनो पक्षों में गुणांकों की तुलना करने पर:-

$4A = 4$

$A = 1$

$A + B = 1$

$B = 0$

माना $2x^2 + x - 3 = t$

$(4x + 1)\,dx = dt$

$$I = \int \frac{4x+1}{\sqrt{2x^2+x-3}}\,dx$$

$$I = \int \frac{1}{\sqrt{t}}\,dt$$

$$I = 2\sqrt{t} + C$$

$$I = 2\sqrt{2x^2+x-3} + C$$

17. $\dfrac{x+2}{\sqrt{x^2-1}}$

उत्तर- माना $x + 2 = A\dfrac{d}{dx}(x^2 - 1) + B \ldots (1)$

$x + 2 = A(2x + B)$

दोनों पक्षों में गुणांकों की तुलना करने पर

$2A = 1$

$A = \frac{1}{2}$

समी. 1 से, $(x+2) = \frac{1}{2}(2x) + 2$

तब, $\int \frac{x+2}{\sqrt{x^2-1}}\,dx$

$= \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}\,dx$

$= \frac{1}{2}\int \frac{(2x)}{\sqrt{x^2-1}}\,dx + \int \frac{2}{\sqrt{x^2-1}}\,dx$ ...(2)

$\Rightarrow \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,dx$ के लिए

माना $x^2 - 1 = t$

$2x\,dx = dt$

$\frac{1}{2}\int \frac{(2x)}{\sqrt{x^2-1}}\,dx$

$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\left[2\sqrt{t}\right]$

$= \sqrt{t} = \sqrt{x^2-1}$

$\int \frac{2}{\sqrt{x^2-1}}\,dx = 2\int \frac{1}{\sqrt{x^2-1}}\,dx$

$= 2\log\left|x+\sqrt{x^2-1}\right|$

समी. 2 से

$I = \int \frac{x+2}{\sqrt{x^2-1}}\,dx$

$I = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + c$

18. $\frac{5x-2}{1+2x+3x^2}$

उत्तर- माना $5x-2 = A\frac{d}{dx}(1+2x+3x^2)+B$

$5x-2 = A(2+6x)+B$

दोनों पक्षों में गुणांकों की तुलना करने पर,

$5 = 6A$

$A = \frac{5}{6}$

$$2A + B = -2$$

$$B = -\frac{11}{3}$$

$$5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$I = \int \frac{5x - 2}{1 + 2x + 3x^2}\, dx$$

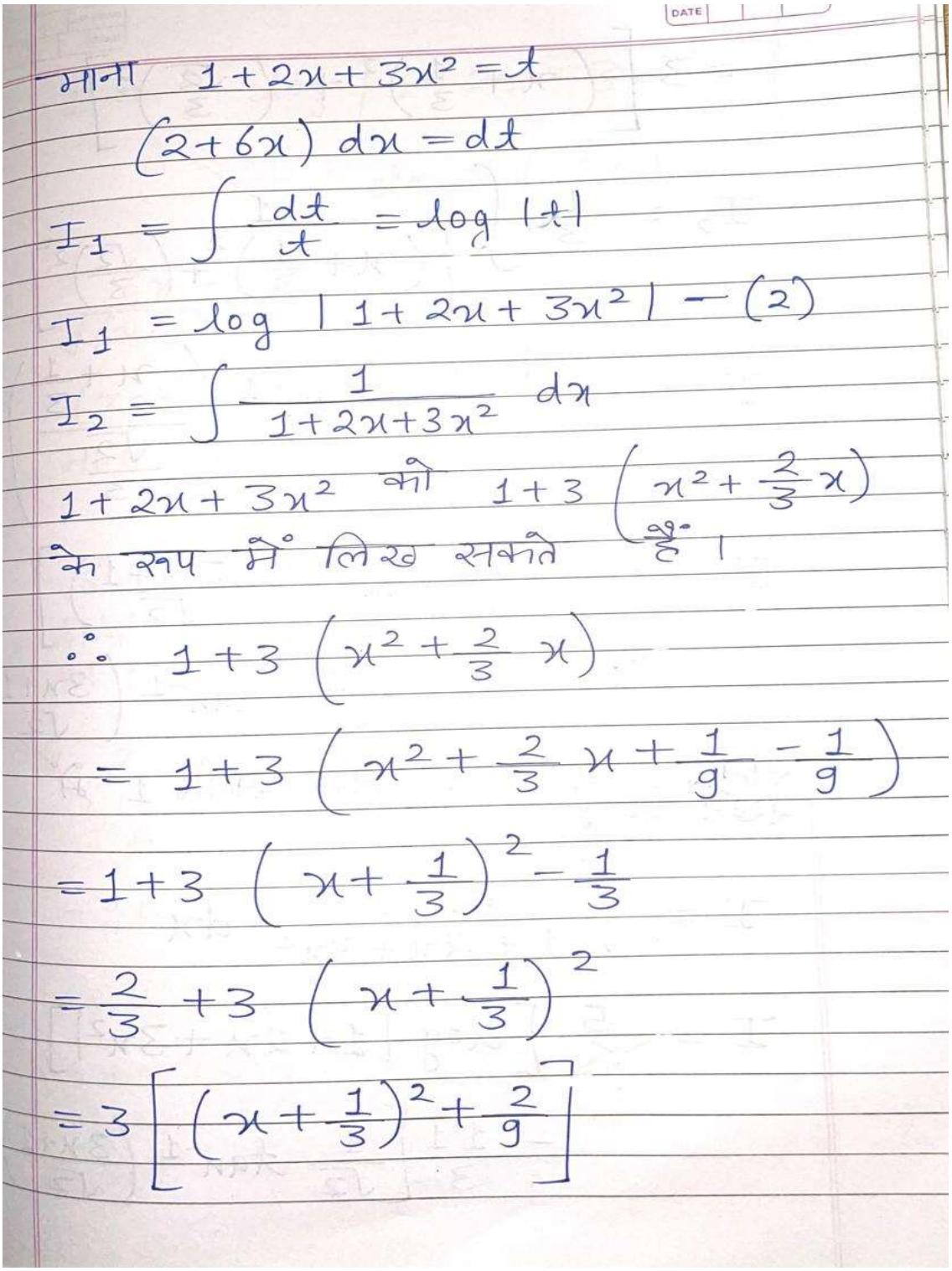
$$I = \int \frac{\frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)}{1 + 2x + 3x^2}\, dx$$

$$I = \frac{5}{6}\int \frac{(2 + 6x)}{1 + 2x + 3x^2}\, dx - \frac{11}{3}\int \frac{1}{1 + 2x + 3x^2}\, dx$$

माना $I_1 = \int \frac{(2 + 6x)}{1 + 2x + 3x^2}$

$$I_2 = \int \frac{1}{1 + 2x + 3x^2}\, dx$$

$$I = \int \frac{5x - 2}{1 + 2x + 3x^2}\, dx = \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad \ldots (1)$$



माना $1+2x+3x^2 = t$

$(2+6x)\,dx = dt$

$I_1 = \int \frac{dt}{t} = \log|t|$

$I_1 = \log|1+2x+3x^2| \quad -(2)$

$I_2 = \int \frac{1}{1+2x+3x^2}\,dx$

$1+2x+3x^2$ को $1+3\left(x^2+\frac{2}{3}x\right)$ के रूप में लिख सकते हैं।

$\therefore 1+3\left(x^2+\frac{2}{3}x\right)$

$= 1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$

$= 1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$

$= \frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$

$= 3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$

$$= 3\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}$$

$$I_2 = \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$I = \frac{1}{3}\left[\frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$$

समी॰ 2 व 3 से मान समी॰ 1 में रखने पर :-

$$I = \int \frac{5x-2}{1+2x+3x^2}\,dx$$

$$I = \frac{5}{6}\left[\log|1+2x+3x^2|\right]$$

$$-\frac{11}{3}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$I = \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

19. $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$

उत्तर- $(x-5)(x-4) = x^2-9x+20$

माना $6x+7 = A\dfrac{d}{dx}(x^2-9x+20)+B$

$6x+7 = A(2x-9)+B$

दोनों पक्षों में गुणांकों की तुलना करने पर :-

$2A = 6$

$A = 3$

$-9A+B = 7$

$B = 34$

$6x+7 = 3(2x-9)+34$

$$I = \int \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$I = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}\,dx$$

$$I = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}\,dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}\,dx$$

माना $I_1 \int \frac{2x-9}{\sqrt{x^2-9x+20}}\,dx$

और

$$I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}\,dx$$

$$I = \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \quad \ldots(1)$$

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}\,dx$$

माना $x^2-9x+20 = t$

$$(2x-9)\,dx = dt$$

$$I_1 = \int \frac{1}{\sqrt{t}}\,dt$$

$$I_1 = 2\sqrt{t}$$

$I_1 = 2\sqrt{x^2-9x+20}$ ... (2)

और $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}\,dx$

$x^2-9x+20 = x^2-9x+20+\frac{81}{4}-\frac{81}{4}$

$= \left(x-\frac{9}{2}\right)^2 - \frac{1}{4}$

$= \left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$I_2 = \int \frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$

$I_2 = \log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|$ ... (3)

समी. 2 व 3 का मान समी. 1 में रखने पर :-

$I = \frac{6x+7}{\sqrt{x^2-9x+20}}\,dx$

$I = 3\left[2\sqrt{x^2-9x+20}\right] + 34 \log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x-20}\right| + c$

$I = 6\left[\sqrt{x^2-9x+20}\right] + 34 \log\left|\left(x-\frac{9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$

24. $\int \frac{dx}{x^2+2x+2}$ बराबर है :

(A) $x \tan^{-1}(x+1) + C$

(B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1} x + C$

(D) $\tan^{-1} x + C$

उत्तर- $I = \int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$

$= \int \frac{dx}{(x+1)^2+1} dx$

$= \left[\tan^{-1}(x+1)\right] + C$

अतः विकल्प (B) सही है।

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ बराबर है :-

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{8}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

उत्तर- $I=\int\frac{1}{\sqrt{9x-4x^2}}dx$

$$I=\int\frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$I=\int\frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}}dx$$

$$I=\int\frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$I=\frac{1}{2}\int\frac{1}{\sqrt{\left[\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2\right]}}dx$$

$$I = \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right] + C$$

$$I = \frac{1}{2}\left[\sin^{-1}\left(\frac{8x-9}{9}\right)\right] + C$$

अतः विकल्प (B) सही है।

Form of Rational	Partial Fraction
1. $\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2. $\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4. $\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5. $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

1. $\frac{x}{(x+1)(x+2)}$

उत्तर- $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$= \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

$x = A(x+2) + B(x+1)$

$A+B = 1 \;,\; A = -1$
$2A+B = 0 \;,\; B = 2$

$\frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$

$I = \int \frac{x}{(x+1)(x+2)}\,dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)}\,dx$

$= -\log|x+1| + 2\log|x+2| + c$

$= \log(x+2)^2 - \log|x+1| + c$

$= \log \frac{(x+2)^2}{(x+1)} + c$

2. $\frac{1}{x^2-9}$

उत्तर- $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$= \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$

$1 = A(x-3)+B(x+3)$

$A+B=0, \quad -3A+3B=1$

$A = -1/6, \quad B = 1/6$

$\frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$

$I = \int \frac{1}{x^2-9}\,dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)}\right) dx$

$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$

$= -\frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right| + C$

3. $\frac{3x-1}{(x-1)(x-2)(x-3)}$

उत्तर- $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$= \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots(1)$$

$A+B+C=0, \; -5A-4B-3C=3, \; 6A+3B+2C=-1$

$A=1, \; B=-5, \; C=4$

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$I = \int \frac{3x-1}{(x-1)(x-2)(x-3)}$$

$$I = \int \left[\frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}\right] dx$$

$$I = \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

4. $\dfrac{x}{(x-1)(x-2)(x-3)}$

उत्तर-

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots(1)$$

$A+B+C=0, -5A-4B-3C=1, 6A+4B+2C=0$

$A=\frac{1}{2}, \quad B=-2, \quad C=\frac{3}{2}$

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$I = \int \frac{x}{(x-1)(x-2)(x-3)}\,dx$$

$$I = \int \left[\frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}\right] dx$$

$$I = \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C.$$

5. $\frac{2x}{x^2+3x+2}$

उत्तर-

$$\frac{2x}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$= \frac{A(x+2)+B(x+1)}{x^2+3x+2}$$

$$2x = A(x+2) + B(x+1) \quad \ldots (1)$$

$A + B = 2 \quad , \quad 2A + B = 0$

$A = -2 \quad , \quad B = 4$

$$\frac{2x}{x^2 + 3x + 2} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$$

$$I = \int \frac{2x}{(x-1)(x-2)(x-3)} \, dx$$

$$I = \int \left[\frac{4}{(x+2)} - \frac{2}{(x+1)} \right] dx$$

$$I = 4 \log |x+2| - 2 \log |x+1| + C$$

6. $\dfrac{1 - x^2}{x(1-2x)}$

उत्तर- अंश $(1 - x^2)$ को हर $x(1-2x)$ से भाग देने पर :-

$$\frac{1 - x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2} \left(\frac{2-x}{x(1-2x)} \right)$$

$$\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$$

$$= (2-x) = A(1-2x) + Bx \quad \ldots (1)$$

$$-2A + B = -1 \quad , \quad A = 2 \quad , \quad B = 3.$$

$$\frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{(1-2x)}$$

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{(1-2x)}\right)$$

$$I = \int \frac{1-x^2}{x(1-2x)}\,dx$$

$$I = \int \left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{(1-2x)}\right)dx\right\}$$

$$I = \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2x| + c$$

$$I = \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + c.$$

8. $\dfrac{x}{(x-1)^2(x+2)}$

उत्तर-

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$A + C = 0,\quad A + B - 2C = 1,\quad -2A + 2B + C = 0$$

$$A = \frac{2}{9},\quad B = \frac{1}{3},\quad C = \frac{-2}{9}$$

$$\frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{-2}{9(x+2)}$$

$$I = \int \frac{x}{(x-1)^2(x+2)}\,dx$$

$$I = \frac{2}{9}\int \frac{1}{(x-1)}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx - \frac{2}{9}\int \frac{1}{(x+2)}\,dx$$

$$I = \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + c$$

$$I = \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c$$

9. $\dfrac{3x+5}{x^3-x^2-x+1}$

उत्तर- $\dfrac{3x+5}{x^3-x^2-x+1} = \dfrac{3x+5}{(x-1)^2(x+1)}$

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$3x+5 = A(x^2-1) + B(x+1) + C(x^2+1-2x) \quad \ldots(1)$$

$$A+C=0,\ B-2C=3,\ -A+B+C=5$$

$$A = -\frac{1}{2},\ B=4,\ C=\frac{1}{2}$$

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$I = \int \frac{3x+5}{(x-1)^2(x+1)}\,dx$$

$$I = \frac{-1}{2}\int \frac{1}{(x-1)}\,dx + 4\int \frac{4}{(x-1)^2}\,dx + \frac{1}{2}\int \frac{1}{(x+1)}\,dx$$

$$I = -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$$I = \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

10. $\frac{2x-3}{(x^2-1)(2x+3)}$

उत्तर- $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$

$$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$$

$$(2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$(2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

$2A + 2B + C = 0$, $A + 5B = 2$, $-3A + 3B - C = -3$

$A = \frac{5}{2}$, $B = \frac{1}{10}$, $C = \frac{-24}{5}$

$$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$I = \int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$

$$I = \frac{5}{2}\int \frac{1}{(x+1)}\,dx - \frac{1}{10}\int \frac{1}{(x-1)}\,dx - \frac{24}{5}\int \frac{1}{(2x+3)}\,dx$$

$$I = \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3| + c$$

$$I = \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + c$$

11. $\dfrac{5x}{(x+1)(x^2-4)}$

उत्तर-

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

$$\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \ldots(1)$$

$A+B+C=0, \quad -B+3C=5, \quad -4A-2B+2C=0$

$A=\frac{5}{3}, \quad B=-\frac{5}{2}, \quad C=-\frac{5}{6}$

$$\frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} - \frac{5}{6(x-2)}$$

$$I = \frac{5}{3}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+2)}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx$$

$$I = \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

12. $\frac{x^3+x+1}{x^2-1}$

उत्तर- अंश (x^3+x+1) को हर (x^2-1) से भाग देने पर :-

$$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}$$

$$\frac{2x+1}{x^2-1} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

$$2x+1 = A(x+1) + B(x-1) \quad \ldots(1)$$

$$A+B=2, \; -A+B=1$$

$$A = \frac{1}{2}, \; B = \frac{3}{2}$$

$$\frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x-1)} + \frac{3}{2(x+1)}$$

$$I = \int \frac{x^3+x+1}{x^2-1}\,dx$$

$$I = \int x\,dx + \frac{1}{2}\int \frac{1}{(x-1)}\,dx + \frac{3}{2}\int \frac{1}{(x+1)}\,dx$$

$$I = \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C.$$

13. $\dfrac{2}{(1-x)(1+x^2)}$

उत्तर-

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

$$A-B=0, \; B-C=0, \; A+C=2$$

$A = 1, B = 1, C = 1$

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{(1-x)} + \frac{x+1}{(1+x^2)}$$

$$I = \int \frac{2}{(1-x)(1+x^2)}\,dx$$

$$I = \int \frac{1}{(1-x)}\,dx + \int \frac{x}{(1+x^2)}\,dx + \int \frac{1}{(1+x^2)}\,dx$$

$$I = -\int \frac{1}{(x-1)}\,dx + \frac{1}{2}\int \frac{2x}{(1+x^2)}\,dx + \int \frac{1}{(1+x^2)}\,dx$$

$$I = -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C$$

14. $\dfrac{3x-1}{(x+2)^2}$

उत्तर-

$$\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$3x-1 = A(x+2) + B$$

$A = 3,\ 2A + B = -1,\ B = -7$

$$\frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$I = \int \frac{3x-1}{(x+2)^2}\,dx$$

$$I = 3\int \frac{1}{(x+2)}\,dx - 7\int \frac{1}{(x+2)^2}\,dx$$

$$I = 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$I = 3\log|x+2| + \frac{7}{(x+2)} + C$$

15. $\frac{1}{x^4-1}$

उत्तर- $\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(x^2+1)}$

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

$A+B+C=0$,
$-A+B+D=0$,
$A+B-C=0$,
$-A+B-D=1$.

$$A = \frac{-1}{4}, B = \frac{1}{4}, C=0, D=\frac{-1}{2}$$

$$\frac{1}{x^4-1} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}$$

$$I = \int \frac{1}{x^4-1}\,dx$$

$$I = \frac{-1}{4}\int \frac{1}{(x+1)}\,dx + \frac{1}{4}\int \frac{1}{(x-1)}\,dx + \left(-\frac{1}{2}\int \frac{1}{(x^2+1)}\,dx\right.$$

$$I = \int \frac{1}{x^4-1}\,dx$$

$$I = -\frac{1}{4}\log|x+1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{-1}x + C$$

$$I = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + c$$

16. $\dfrac{1}{x(x^n+1)}$

[संकेत : अंश एवं हर को x^{n-1} से गुणा कीजिए और $x^n = t$ रखिए।]

उत्तर- $\dfrac{1}{x(x^n+1)} = \dfrac{x^{n-1}}{x^{n-1}x(x^n+1)} = \dfrac{x^{n-1}}{x^n(x^n+1)}$

माना $x^n = t$

$$nx^{n-1}dx = dt$$

$$I = \int \frac{1}{x(x^n+1)}dx$$

$$I = \int \frac{x^{n-1}}{x^n(x^n+1)}dx$$

$$I = \frac{1}{n}\int \frac{1}{t(t+1)}dt$$

माना $\dfrac{1}{t(t+1)} = \dfrac{A}{t} + \dfrac{B}{(t+1)}$

$$1 = A(t+1) + Bt \quad \ldots (1)$$

$A = 1, B = -1.$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$$

$$I = \int \frac{1}{x(x^n+1)}\,dx$$

$$I = \frac{1}{n}\int \left\{\frac{1}{t} - \frac{1}{(1+t)}\right\} dt$$

$$I = \frac{1}{n}\left[\log|t| - \log|t+1|\right] + c$$

$$I = \frac{-1}{n}\left[\log|x^n| - \log|x^n+1|\right] + c$$

$$I = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + c$$

17. $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$

[संकेत : $\sin x = t$ रखिए]

उत्तर-

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

माना $\sin x = t$

$\cos x\,dx = dt$

$$I = \int \frac{\cos x}{(1-\sin x)(2-\sin x)}\, dx$$

$$I = \int \frac{dt}{(1-t)(2-t)}$$

माना : $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$$1 = A(2-t) + B(1-t) \quad \ldots (1)$$

$A = \frac{-1}{4}$, $B = \frac{1}{4}$, $C = 0$, $D = -\frac{1}{2}$

$$\frac{1}{x^4-1} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1}$$

$$I = \int \frac{1}{x^4-1}\, dx$$

$$I = -\frac{1}{4}\int \frac{1}{(x+1)}\, dx + \frac{1}{4}\int \frac{1}{(x-1)}\, dx +$$

$$\left(-\frac{1}{2}\int \frac{1}{(x^2+1)}\, dx\right)$$

$$I = \int \frac{1}{x^4-1}\, dx$$

$$I = -\frac{1}{4}\log|x+1) + \frac{1}{4}\log|x-1|$$

$$-\frac{1}{2}\tan^{-1}x + c$$

$$I = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + c$$

19. $\dfrac{2x}{(x^2+1)(x^2+3)}$

उत्तर- माना $x^2 = t$

$$2x\,dx = dt$$

$$I = \int \frac{2x}{(x^2+1)(x^2+3)}\,dx$$

$$I = \int \frac{dt}{(t+1)(t+3)} \quad \ldots(1)$$

माना $\dfrac{1}{(t+1)(t+3)} = \dfrac{A}{(t+1)} + \dfrac{B}{(t+3)}$

$$1 = A(t+3) + B(t+1) \quad \ldots(2)$$

$A + B = 0$, $3A + B = 1$

$A = \dfrac{1}{2}$, $B = \dfrac{-1}{2}$

$$\frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{2}{2(t+3)}$$

$$I = \int \frac{2x}{(x^2+1)(x^2+3)}\,dx$$

$$I = \int \left\{ \frac{1}{2(t+1)} - \frac{2}{2(t+3)} \right\} dt$$

$$I = \frac{1}{2}\log|t+1| - \frac{1}{2}\log|t+3| + c$$

$$I = \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + c$$

$$I = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c$$

20. $\dfrac{1}{x(x^4-1)}$

उत्तर- अंश और हर को x^3 से गुणा करने पर :-

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

$$I = \int \frac{1}{x(x^4-1)}\,dx = \int \frac{x^3}{x^4(x^4-1)}\,dx$$

माना $x^4 = t$

$$4x^3\,dx = dt$$

$$I = \int \frac{1}{x(x^4-1)}\,dx$$

$$I = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

माना $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$

$1 = A(t-1) + Bt \quad \ldots (1)$

$A = -1, B = 1$

$$\frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{(t-1)}$$

$$I = \int \frac{1}{x(x^4-1)}\,dx$$

$$I = \frac{1}{4}\int \left\{\frac{-1}{t} + \frac{1}{(t-1)}\right\} dt$$

$$I = \frac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$$

$$I = \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C$$

$$I = \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$$

21. $\frac{1}{(e^x-1)}$

[संकेत : $e^x = t$ रखिए]

उत्तर- माना $e^x = t$

$e^x dx = dt$

$$I = \int \frac{1}{(e^x - 1)} dx$$

$$I = \int \frac{1}{(t-1)} \times \frac{dt}{t}$$

$$I = \int \frac{dt}{t(t-1)}$$

माना $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$

$1 = A(t-1) + Bt \quad \ldots (1)$

$A = -1, B = 1$

$$\frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{(t-1)}$$

$$I = \int \frac{1}{t(t-1)} dt$$

$$I = \log \left| \frac{t-1}{t} \right| + C$$

$$I = \log\left|\frac{e^x - 1}{e^x}\right| + C$$

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ बराबर है :

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

(B) $\log\left|\frac{(x-2)^2}{x-2}\right| + C$

(C) $\log\left|\left(\frac{(x-1)}{(x-2)}\right)^2\right| + C$

(D) $\log|(x-1)(x-2)| + C$

उत्तर- माना $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1)$

$A + B = 1, -2A - B = 0$

$A = -1, B = 2$

$\frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{1}{(x-2)}$

$$I = \int \frac{x}{(x-1)(x-2)}\,dx$$

$$I = \int \left\{ \frac{-1}{(x-1)} + \frac{1}{(x-2)} \right\} dx$$

$$I = -\log|x-1| + 2\log|x-2| + c$$

$$I = \log\left|\frac{(x-2)^2}{x-1}\right| + c$$

अतः विकल्प (B) सही है।

23. $\int \frac{dx}{x(x^2+1)}$ बराबर है :

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + c$

(B) $\log|x| + \frac{1}{2}\log(x^2+1) + c$

(C) $-\log|x| + \frac{1}{2}\log(x^2+1) + c$

(D) $\frac{1}{2}\log|x| + \log(x^2+1) + c$

उत्तर- $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$

$$1 = A(x^2+1) + (Bx+C)x$$

$$A + B = 0,\ C = 0,\ A = 1,\ B = -1$$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \left(\frac{-x}{(x^2+1)}\right)$$

$$I = \int \frac{1}{x(x^2+1)}\,dx$$

$$I = \int \left\{ \frac{1}{x} + \frac{(-x)}{(x^2+1)} \right\} dx$$

$$I = \log|x| - \frac{1}{2}\log|x^2+1| + c$$

अतः विकल्प (A) सही है।

खंडशः समाकलन

> $\int f(x).g(x).dx = f(x)\int g(x)\,dx - \int\left[\int g(x)dx\, f'(x)\right]dx$

OR

> $\int u.v\,dx = u\int v\,dx - \int\left[\frac{d}{dx}(u)\int v\,dx\right]dx$

प्रश्नावली 7.6

1. $x \sin x$

उत्तर- $I = \int x \sin x\,dx$

$I = x\int \sin x\,dx - \int\left\{\frac{d}{dx}(x)\int \sin x\right\}dx$

$I = x(-\cos x) - \int 1.(-\cos x)\,dx$

$I = -x\cos x + \sin x + c$

2. $x \sin 3x$

उत्तर- $I = \int x \sin 3x\,dx$

$I = x\int \sin 3x\,dx - \int\left\{\frac{d}{dx}(x)\int \sin 3x.dx\right\}$

$I = x\left(\frac{-\cos 3x}{3}\right) - \int 1.\left(\frac{-\cos 3x}{3}\right)dx$

$$I = -\frac{x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\, dx$$

$$I = -\frac{x\cos 3x}{3} + \frac{1}{9}\sin 3x + c$$

3. $x^2 e^x$

उत्तर- $I = \int x^2 e^x\, dx$

$$I = x^2\int e^x dx - \int\left\{\frac{d}{dx}(x^2)\int e^x dx\right\}dx$$

$$I = x^2e^x - \int 2x\cdot e^x dx$$

$$I = x^2e^x - 2\int x\, e^x dx$$

$$I = x^2e^x - 2\left[x\cdot\int e^x dx - \int\left\{\frac{d}{dx}(x^2)\int e^x dx\right\}dx\right]$$

$$I = x^2e^x - 2\left[xe^x - \int e^x dx\right]$$

$$I = x^2e^x - 2[xe^x - e^x]$$

$$I = x^2e^x - 2xe^x + 2e^x + c$$

$$I = e^x(x^2 - 2x + 2) + c$$

4. $x \log x$

उत्तर- $I = \int x \log x\, dx$

$$I = \log x \int x\,dx - \int \left\{ \frac{d}{dx}(\log x) \int x\,dx \right\} dx$$

$$I = \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx$$

$$I = \frac{x^2 \log 2x}{2} - \int \frac{x}{2}\,dx$$

$$I = \frac{x^2 \log 2x}{2} - \frac{x^4}{4} + C$$

5. $x \log 2x$

उत्तर- $I = \int x \log 2x\,dx$

$$I = \log 2x \int x\,dx - \int \left\{ \frac{d}{dx}(\log 2x) \int x\,dx \right\} dx$$

$$I = \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2}\,dx$$

$$I = \frac{x^2 \log 2x}{2} - \int \frac{x}{2}\,dx$$

$$I = \frac{x^2 \log 2x}{2} - \frac{x^4}{4} + C$$

6. $x^2 \log x$

उत्तर- $I = \int x^2 \log x \, dx$

$I = \log x \int x^2 dx - \int \left\{ \frac{d}{dx}(\log x) \int x^2 dx \right\} dx$

$I = \log x \left(\frac{x^3}{3}\right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$

$I = \frac{x^3 \log x}{3} - \frac{1}{3}\int x^2 dx$

$I = \frac{x^3 \log x}{3} - \frac{x^3}{9} + c$

7. $x \sin^{-1} x$

उत्तर- $I = \int x \sin^{-1} x \, dx$

$I = \sin^{-1} x \int x \, dx - \int \left\{ \frac{d}{dx}(\sin^{-1} x) \int x \, dx \right\} dx$

$I = \sin^{-1} x \left(\frac{x^2}{2}\right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$

$I = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}} dx$

$I = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$

$$I = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$I = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left\{ \int \sqrt{1-x^2}\, dx - \int \frac{1}{\sqrt{1-x^2}} dx \right\}$$

$$I = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left\{ \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x - \sin^{-1} x \right\} + c$$

$$I = \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x - \sin^{-1} x + c$$

$$I = \frac{1}{4}(2x^2 - 1)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$$

8. $x \tan^{-1} x$

उत्तर- $I = \int x \tan^{-1} x \, dx$

$$I = \tan^{-1} x \int x\, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) \int x\, dx \right\} dx$$

$$I = \tan^{-1}x\left(\frac{x^2}{2}\right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\,dx$$

$$I = \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$I = \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\left(x - \tan^{-1}x\right)$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$$

9. $x\cos^{-1}x$

उत्तर- $I = \int x\cos^{-1}x\,dx$

$$I = \cos^{-1}x\int x\,dx - \int\left\{\frac{d}{dx}(\cos^{-1}x)\int x\,dx\right\}dx$$

$$I = \cos^{-1}x\left(\frac{x^2}{2}\right) - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2}\,dx$$

$$I = \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$I = \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$I = \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2}\int \left\{\sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}}\right)\right\} dx$$

$$I = \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\int \left(\frac{-1}{\sqrt{1-x^2}}\right) dx$$

$$I = \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2}\cos^{-1} x \quad \ldots(1)$$

जहाँ $I_1 = \int \sqrt{1-x^2} \cdot 1\, dx$

$$I_1 = x\sqrt{1-x^2} - \int \frac{d}{dx}\sqrt{1-x^2} \int x\, dx$$

$$I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{\sqrt{1-x^2}} x\, dx$$

$$I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} x\, dx$$

$$I_1 = x\sqrt{1-x^2} - \left\{\int \sqrt{1-x^2}\, dx + \int \frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$I_1 = x\sqrt{1-x^2} - \{I_1 + \cos^{-1}x\}$$

$$2I_1 = x\sqrt{1-x^2} - \cos^{-1}x$$

$$I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x$$

समी. 1 में रखने पर :-

$$I = \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x\right) - \frac{1}{2}\cos^{-1}x$$

$$I = \frac{(2x^2-1)}{4}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + c$$

10. $(\sin^{-1}x)^2$

उत्तर- $I = \int (\sin^{-1}x)^2 \cdot 1\, dx$

$$I = (\sin^{-1}x)^2 \int 1\, dx - \int \left\{\frac{d}{dx}(\sin^{-1}x)^2 \cdot \int 1\, dx\right\} dx$$

$$I = (\sin^{-1}x)^2 \cdot x - \int \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \cdot x\, dx$$

$$I = x(\sin^{-1}x)^2 + \int \sin^{-1}x \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$$

$$I = x(\sin^{-1}x)^2 + \left[\sin^{-1}x \int \frac{-2x}{\sqrt{1-x^2}}\,dx - \int \left\{\frac{d}{dx}(\sin^{-1}x)\int \frac{-2x}{\sqrt{1-x^2}}\,dx\right\}dx\right]$$

$$I = x(\sin^{-1}x)^2 + \left[\sin^{-1}x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\,dx\right]$$

$$I = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - \int 2\,dx$$

$$I = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + c$$

11. $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

उत्तर-

$$I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}\,dx$$

$$I = \int \frac{-1}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1}x\,dx$$

$$I = \frac{-1}{2}\left[\cos^{-1}x \int \frac{-2x}{\sqrt{1-x^2}}\,dx - \int \frac{d}{dx}(\cos^{-1}$$

$$\int \left\{ \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \Big]$$

$$I = \frac{-1}{2} \left[\cos^{-1} x \cdot 2\sqrt{1-x^2} + \int \frac{-1}{\sqrt{1-x^2}} \cdot 2 \sqrt{1-x^2}\, dx \right]$$

$$I = \frac{-1}{2} \left[2\sqrt{1-x^2} \cos^{-1} x + \int 2\, dx \right]$$

$$I = \frac{-1}{2} \left[2\sqrt{1-x^2} \cos^{-1} x + 2x \right] + c$$

$$I = -\left[\sqrt{1-x^2} \cos^{-1} + x \right] + c$$

12. $x \sec^2 x$

उत्तर- $I = \int x \sec^2 x\, dx$

$$I = x \int \sec^2 x\, dx - \int \left\{ \frac{d}{dx}(x) \int \sec^2 x\, dx \right\} dx$$

$$I = x \tan x - \int 1 \cdot \tan x\, dx$$

$$I = x \tan x + \log |\cos x| + c$$

13. $\tan^{-1} x$

उत्तर- $I = \int \tan^{-1} x\, dx$

$$I = \tan^{-1} x \int 1\, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) \int 1.\, dx \right\} dx$$

$$I = \tan^{-1} x . x - \int \frac{1}{1+x^2} x\, dx$$

$$I = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$I = x \tan^{-1} x - \frac{1}{2} \log [1+x^2] + c$$

14. $x (\log x)^2$

उत्तर- $I = \int x (\log x)^2 dx$

$$I = (\log x)^2 \int x\, dx - \int \left\{ \frac{d}{dx} (\log x)^2 \int x\, dx \right\} dx$$

$$I = \frac{x^2}{2} (\log x)^2 - \left[\int 2 \log x \frac{1}{x} \cdot \frac{x^2}{2} dx \right]$$

$$I = \frac{x^2}{2} (\log x)^2 - \int x \log x\, dx$$

$$I = \frac{x^2}{2} (\log x)^2 - \left[\log x \int x\, dx - \right.$$

$$\int \left\{ \frac{d}{dx}(\log x) \int x\,dx \right\} dx$$

$$I = \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2} \log x - \int \frac{1}{2} \cdot \frac{x^2}{2} dx \right] = \frac{x^2}{2}(\log)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + c.$$

15. $(x^2+1)\log x$

उत्तर- $I = \int (x^2+1)\log x\,dx$

$I = \int x^2 \log x\,dx + \int \log x\,dx$

माना $I = I_1 + I_2$

$I_1 = \int x^2 \log x\,dx$, $I_2 = \int \log x\,dx$

$I_1 = \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1$

$I_2 = x\log x - x + C_2$

$I = \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$

$I = \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + c.$

16. $e^x(\sin x + \cos x)$

उत्तर- $I = \int e^x (\sin x + \cos x)\, dx$

$f(x) = \sin x$

$f'(x) = \cos x$

$I = \int e^x (\sin x + \cos x)\, dx$

$I = \int e^x \{f(x) + f'(x)\}\, dx$

$\int e^x \{f(x) + f'(x)\}\, dx = e^x f(x) + c$

$I = e^x \sin x + c$

17. $\frac{xe^x}{(1+x)^2}$

उत्तर- $I = \int \frac{xe^x}{(1+x)^2}\, dx = \int e^x \left\{\frac{x}{(1+x)^2}\right\} dx$

$I = \int e^x \left\{\frac{1+x-1}{(1+x)^2}\right\} dx$

$I = \int e^x \left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\} dx$

माना $f(x) = \frac{1}{1+x}$

$$f'(x) = \frac{-1}{(1+x)^2}$$

$$I = \int \frac{xe^x}{(1+x)^2}\,dx$$

$$I = \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$$

$$I = \int e^x \{ f(x) + f'(x) \}\,dx$$

$$\int e^x \{ f(x) + f'(x) \}\,dx = e^x f(x) + c$$

$$I = \int \frac{xe}{(1+x^2)}\,dx = \frac{e^x}{1+x} + c$$

18. $e^x \left(\frac{1 + \sin x}{1 + \cos x} \right)$

उत्तर-

$$= e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right)$$

$$= e^x \frac{\left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2 \cos^2 \frac{x}{2}}$$

$$= \frac{1}{2} e^x \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2$$

$$= \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2$$

$$= \frac{1}{2} e^x \left[1 + \tan \frac{x}{2} \right]^2$$

$$= \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right]$$

$$= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right]$$

$$I = \int \frac{e^x (1 + \sin x)}{(1 + \cos x)} dx$$

$$I = \int e^x \left[\sec^2 x/2 + 2 \tan x/2 \right] dx$$

माना $2 \tan \frac{x}{2} = f(x)$

$$f'(x) = \sec^2 x/2$$

$$I = \int e^x \{ f'(x) + f(x) \} dx$$

$$I = e^x \tan \frac{x}{2} + c$$

19. $e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$

उत्तर- $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$

माना $\frac{1}{x} = f(x)$

$f'(x) = -1/x^2$.

$$I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \frac{e^x}{x} + c$$

20. $\frac{(x-3)}{(x-1)^2}$

उत्तर- $I = \int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx$

$$I = \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx$$

$$I = \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$$

माना $f(x) = \frac{1}{(x-1)^2}$

$f'(x) = \frac{2}{(x-1)^3}$

$I = \int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx$

$I = \frac{e^x}{(x-1)^2} + c$

21. $e^{2x} \sin x$

उत्तर- $I = \int e^{2x} \sin x \, dx$

$I = \sin x \int e^{2x} dx - \int \left\{ \frac{d}{dx}(\sin x) \int e^{2x} dx \right\} dx$

$I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2} dx$

$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$

$I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + c$

$$I = e^{2x}/5\left[2\sin x - \cos x\right] + c$$

22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

उत्तर- माना $x = \tan\theta$

$$dx = \sec^2\theta\, d\theta$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta$$

$$I = \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

$$I = \int 2\theta \sec^2\theta\, d\theta$$

$$I = 2\int \theta \sec^2\theta\, d\theta$$

$$I = 2\left[\theta\int \sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}(\theta)\int \sec^2\theta\, d\theta\right\} d\theta\right]$$

$$I = 2\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]$$

$$I = 2\left[\theta\tan\theta - \log|\cos\theta|\right] + c$$

$$I = 2\left[x \tan^{-1} x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + c$$

$$I = 2x \tan^{-1} x + 2 \log (1+x^2)^{-1/2} + c$$

$$I = 2x \tan^{-1} x + 2\left[-\frac{1}{2} \log (1+x^2)\right] + c$$

$$I = 2x \tan^{-1} x - \log (1+x^2) + c$$

23. $\int x^2 e^{x^3} dx$ बराबर है:

(A) $1/3\ e^{x^3} + c$
(B) $1/3\ e^{x^2} + c$
(C) $1/2\ e^{x^3} + c$
(D) $1/2\ e^{x^2} + c$

उत्तर- $I \int x^2 e^{x^3} dx$

माना $x^3 = t$

$$3x^2 dx = dt$$

$$I = \frac{1}{3} \int (e^t)\, dt$$

$$I = \frac{1}{3} (e^t) + c$$

$$I = \frac{1}{3} e^{x^3} + c$$

अतः विकल्प (A) सही है।

24. $\int e^x \sec x (1 + \tan x)\, dx$ बराबर है:

(A) $e^x \cos x + c$
(B) $e^x \sec x + c$
(C) $e^x \sin x + c$
(D) $e^x \tan x + c$

उत्तर- $I = \int e^x \sec x (1 + \tan x)\, dx$

$I = \int e^x (\sec x + \sec x \tan x)\, dx$

माना $\sec x = f(x)$

$f'(x) = \sec x \tan x$

$\int e^x (f(x) + f'(x))\, dx = e^x f(x) + c$

$I = \int e^x (\sec x + \sec x \tan x)\, dx$

$I = e^x \sec x + c$

अतः विकल्प B सही है।

(1) $\int \sqrt{x^2+a^2}\, dx = \frac{1}{2} x \sqrt{x^2+a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2+a^2}| + c$

(2) $\int \sqrt{a^2-x^2}\, dx = \frac{1}{2} x \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

(3) $\int \sqrt{x^2-a^2}\, dx = \frac{1}{2} x \sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}| + c$

प्रश्नावली 7.7

प्रश्न-1: $\sqrt{4-x^2}$

उत्तर- $I = \int \sqrt{4-x^2}\, dx$

$I = \int \sqrt{(2)^2-(x)^2}\, dx$

$I = \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{a} + c$

$I = \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{a} + c$

प्रश्न-2: $\sqrt{1-4x^2}$

उत्तर- $I = \int \sqrt{1-4x^2}\, dx$

$$I = \int \sqrt{(1)^2 - (2x)^2}\, dx$$

माना $2x = t$

$2dx = dt$

$$I = \frac{1}{2}\int \sqrt{(1)^2 - (t)^2}\, dt$$

$$I = \frac{1}{2}\left[\frac{1}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right] + c$$

$$I = \frac{t}{4}\sqrt{1-t^2} + \frac{1}{4}\sin^{-1} 2x + c$$

$$I = \frac{x}{2}\sqrt{\frac{4}{1} - 4x^2} + \frac{1}{4}\sin^{-1} 2x + c$$

3. $\sqrt{x^2+4x+6}$

उत्तर- $I = \int \sqrt{x^2+4x+6}\, dx$

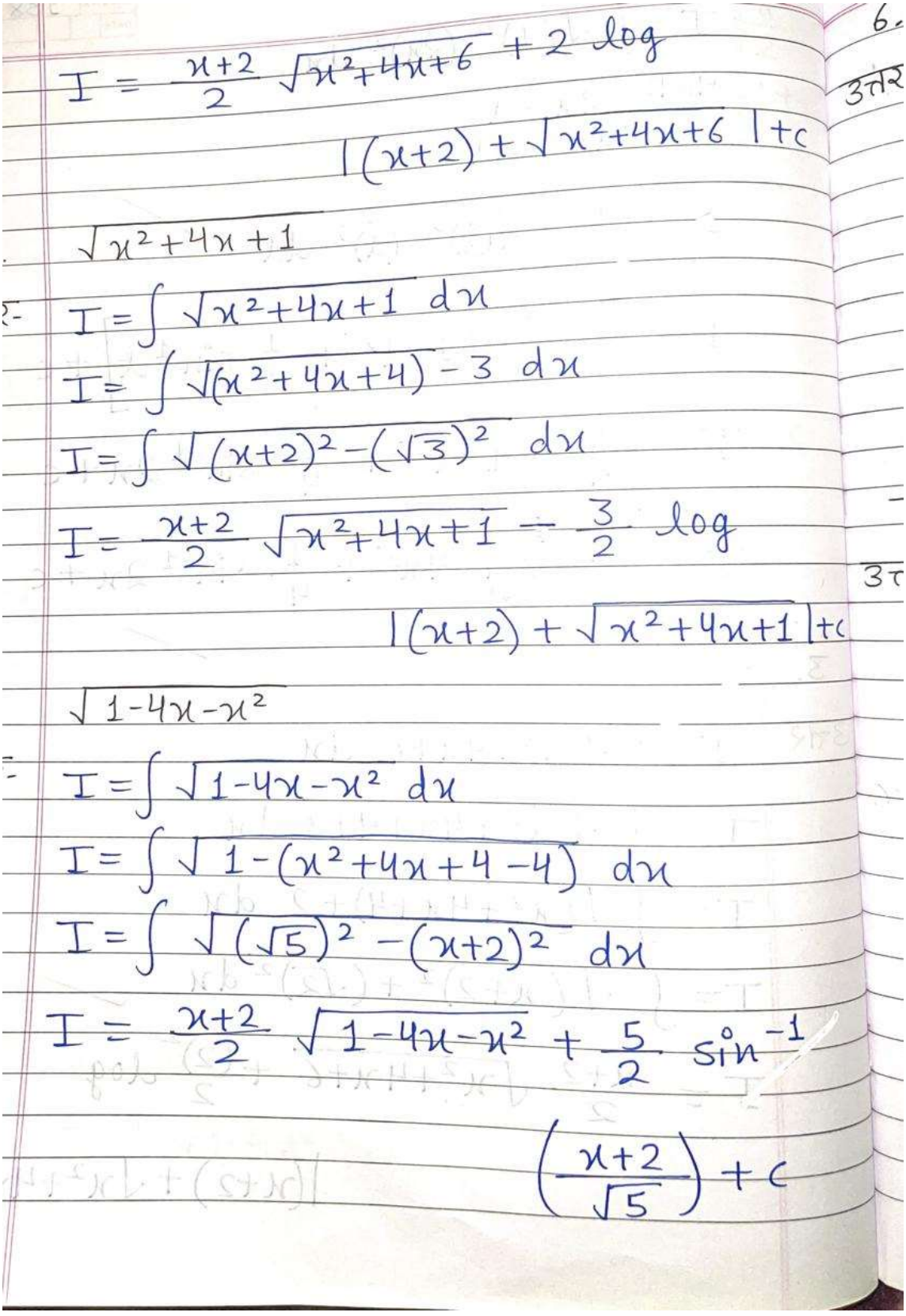
$$I = \int \sqrt{x^2+4x+4+2}\, dx$$

$$I = \int \sqrt{(x^2+4x+4)+2}\, dx$$

$$I = \int \sqrt{(x+2)^2 + (\sqrt{2})^2}\, dx$$

$$I = \frac{x+2}{2}\sqrt{x^2+4x+6} + \frac{(2)^2}{2}\log$$

$$|(x+2) + \sqrt{x^2+4x+6}| + c$$



$$I = \frac{x+2}{2}\sqrt{x^2+4x+6} + 2\log\left|(x+2)+\sqrt{x^2+4x+6}\right| + C$$

$\sqrt{x^2+4x+1}$

$$I = \int \sqrt{x^2+4x+1}\,dx$$

$$I = \int \sqrt{(x^2+4x+4)-3}\,dx$$

$$I = \int \sqrt{(x+2)^2-(\sqrt{3})^2}\,dx$$

$$I = \frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right| + C$$

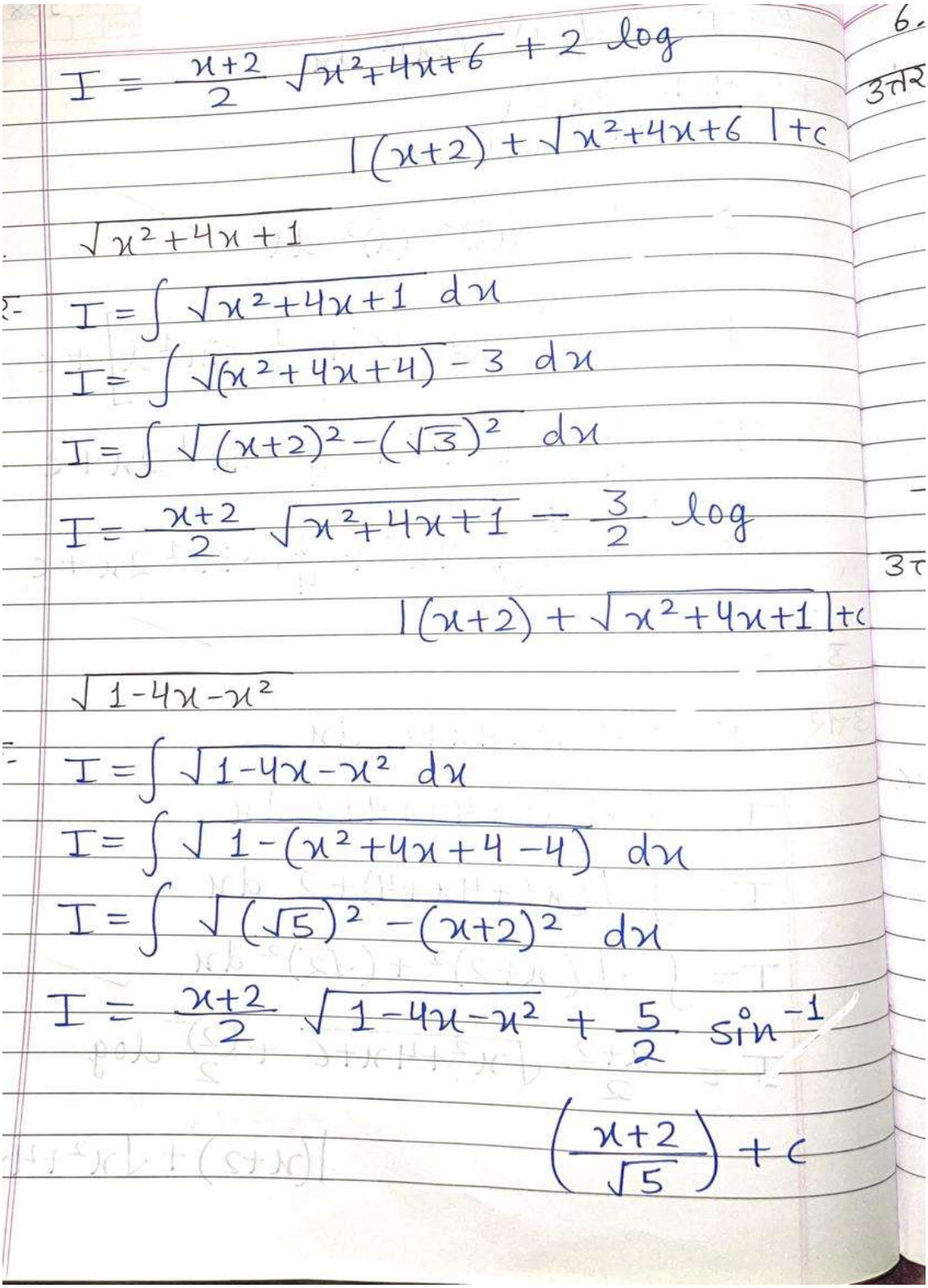
$\sqrt{1-4x-x^2}$

$$I = \int \sqrt{1-4x-x^2}\,dx$$

$$I = \int \sqrt{1-(x^2+4x+4-4)}\,dx$$

$$I = \int \sqrt{(\sqrt{5})^2-(x+2)^2}\,dx$$

$$I = \frac{x+2}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$



$$I = \frac{x+2}{2}\sqrt{x^2+4x+6} + 2\log\left|(x+2)+\sqrt{x^2+4x+6}\right| + c$$

$\sqrt{x^2+4x+1}$

$$I = \int \sqrt{x^2+4x+1}\,dx$$

$$I = \int \sqrt{(x^2+4x+4)-3}\,dx$$

$$I = \int \sqrt{(x+2)^2-(\sqrt{3})^2}\,dx$$

$$I = \frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right| + c$$

$\sqrt{1-4x-x^2}$

$$I = \int \sqrt{1-4x-x^2}\,dx$$

$$I = \int \sqrt{1-(x^2+4x+4-4)}\,dx$$

$$I = \int \sqrt{(\sqrt{5})^2-(x+2)^2}\,dx$$

$$I = \frac{x+2}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

6. $\sqrt{x^2+4x-5}$

उत्तर- $I = \int \sqrt{x^2+4x-5}\, dx$

$I = \int \sqrt{(x^2+4x+4)-9}\, dx$

$I = \int \sqrt{(x+2)^2-(3)^2}\, dx$

$I = \frac{x+2}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log$

$|(x+2)+\sqrt{x^2+4x-5}|+c$

7. $\sqrt{1+3x-x^2}$

उत्तर- $I = \int \sqrt{1+3x-x^2}\, dx$

$I = \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\, dx$

$I = \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\, dx$

$I = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\, dx$

$I = \frac{(x-3/2)}{2}\sqrt{1+3x-x^2} + \frac{13}{4\times 2}$

$\sin^{-1}\left(x-\frac{3}{2}\Big/\frac{\sqrt{13}}{2}\right)+c$

$$I = \frac{(2x-3)}{4}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

8. $\sqrt{x^2+3x}\ dx$

उत्तर- $I = \int \sqrt{x^2+3x}\ dx$

$$I = \int \sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\ dx$$

$$I = \sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\ dx$$

$$I = \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$$

$$I = \frac{(2x+3)}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$$

9. $\sqrt{1+\frac{x^2}{9}}$

उत्तर- $I = \int \sqrt{1+\frac{x^2}{9}}\, dx$

$I = \frac{1}{3}\int \sqrt{9+x^2}\, dx$

$I = \frac{1}{3}\int \sqrt{(3)^2+(x)^2}\, dx$

$I = \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9} - \frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+c$

$I = \frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+c$

10. $\int \sqrt{1+x^2}\, dx$ बराबर है :

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+c$

(B) $\frac{2}{3}(1+x^2)^{3/2}+c$

(C) $\frac{2}{3}x(1+x^2)^{1/2}+c$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+c$

उत्तर- $I = \int \sqrt{1+x^2}\, dx$

$$I = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + c$$

अतः, विकल्प A (सही) है।

11. $\int \sqrt{x^2-8x+7}\,dx$ बराबर है :

(A) $\frac{1}{2}(x-4)\left(\sqrt{x^2-8x+7}\right) + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + c$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + c$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + c$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + c$

उत्तर- $I = \int \sqrt{x^2-8x+7}\,dx$

$I = \int \sqrt{x^2-8x+16-9}\,dx$

$I = \int \sqrt{(x-4)^2-(3)^2}\,dx$

$I = \frac{(x-4)}{2}\sqrt{x^2-8x+7} - \frac{9}{2}\log$

$\left|(x-4)+\sqrt{x^2-8x+7}\right| + c$

अतः विकल्प (D) सही है।

प्रश्नावली - 7.9

1. $\int_0^1 \frac{x}{x^2+1}\,dx$

उत्तर- माना $x^2+1=t$

$2x\,dx=dt$

जब $x=0,\ t=1.$

और जब $x=1,\ t=2.$

$$I=\int_0^1 \frac{x}{x^2+1}\,dx$$

$$I=\frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$I=\frac{1}{2}\left[\log|t|\right]_1^2$$

$$I=\frac{1}{2}\left[\log 2-\log 1\right]$$

$$I=\frac{1}{2}\log 2.$$

2. $\int_0^{\pi/2} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$

उत्तर- माना $\sin\phi=t$

$\cos\phi\, d\phi = dt$

जब $\phi = 0, t = 0$

और जब $\phi = \pi/2, t = 1$

$$I = \int_0^{\pi/2} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$$

$$I = \int_0^{\pi/2} \sqrt{\sin\phi}\, \cos^4\phi \cos\phi\, d\phi$$

$$I = \int_0^1 \sqrt{t}\, (1-t^2)^2\, dt$$

$$I = \int_0^1 t^{1/2} (1 + t^4 - 2t^2)\, dt$$

$$I = \int_0^1 \left[t^{1/2} + t^{9/2} - 2t^{5/2} \right] dt$$

$$I = \left[\frac{t^{3/2}}{\frac{3}{2}} + \frac{t^{11/2}}{\frac{11}{2}} - \frac{t^{7/2}}{\frac{7}{2}} \right]_0^1$$

$$I = \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$I = \frac{154 + 42 - 132}{231}$$

$$I = \frac{64}{231}$$

3. $$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

उत्तर- $$I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

माना $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

जब $x = 0, \theta = 0$ और जब $x = 1, \theta = \pi/4$

$$I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \sec^2\theta \, d\theta$$

$$I = \int_0^{\pi/4} \sin^{-1}(\sin 2\theta) \sec^2\theta \, d\theta$$

$$I = \int_0^{\pi/4} 2\theta \cdot \sec^2\theta \, d\theta$$

$$I = 2\int_0^{\pi/4} \theta \cdot \sec^2\theta \, d\theta$$

$$I = 2\left[\theta \int \sec^2\theta\, d\theta \int \left\{\left(\frac{d}{dx}\theta\right) \int \sec^2\theta\, d\theta\right\} d\theta\right]_0^{\pi/4}$$

$$I = 2\left[\theta \tan\theta - \int \tan\theta\, d\theta\right]_0^{\pi/4}$$

$$I = 2\left[\theta \tan\theta - \log|\cos\theta|\right]_0^{\pi/4}$$

$$I = 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$I = 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$I = 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$I = \frac{\pi}{2} - \log 2$$

4. $\int_0^2 x\sqrt{x+2}$ $(x+2 = t^2$ रखिए$)$

उत्तर- $\int_0^2 x\sqrt{x+2}\, dx$

माना $x+2 = t^2$

$dx = 2t\,dt$

जब $x=0,\ t=\sqrt{2}$ और जब $x=2,\ t=2$

$$I = \int_0^2 x\sqrt{x+2}\,dx$$

$$I = \int_{\sqrt{2}}^{t} (t^2-2)\sqrt{t^2}\,2t\,dt$$

$$I = 2\int_{\sqrt{2}}^{2} (t^2-2)\,t^2\,dt$$

$$I = 2\int_{\sqrt{2}}^{2} (t^4-2t^2)\,dt$$

$$I = 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^{2}$$

$$I = 2\left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right]$$

$$I = 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$

$$I = 2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$I = \frac{16(2+\sqrt{2})}{15}$$

$$I = \frac{16\sqrt{2}\,(\sqrt{2}+1)}{15}$$

5. $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$

उत्तर- माना $\cos x = t$

$-\sin x\,dx = dt$

जब $x=0,\ t=1$ और जब $x=\pi/2,\ t=0$.

$$I = \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$I = -\int_1^0 \frac{dt}{1+t^2}$$

$$I = -\left[\tan^{-1} t\right]_1^0$$

$$I = -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$I = -\left[-\frac{\pi}{4}\right]$$

$$I = \frac{\pi}{4}.$$

6. $\int_0^2 \frac{dx}{x+4-x^2}$

उत्तर- $I = \int_0^2 \frac{dx}{x+4-x^2}$

$I = \int_0^2 \frac{dx}{-(x^2-x-4)}$

$I = \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$

$I = \int_0^2 \frac{dx}{\left[-\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$

$I = \int_0^2 \frac{dx}{\left[\left(\sqrt{17}/2\right)^2-\left(x-\frac{1}{2}\right)^2\right]}$

माना $x - 1/2 = t$

$dx = dt$

जब $x=0$, $t=-1/2$ और जब $x=2$, $t=$

$I = \int_{-1/2}^{3/2} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$

$$I = \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log\left(\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right)\right]_{-1/2}^{3/2}$$

$$I = \frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \log\frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$I = \frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3} - \log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$I = \frac{1}{\sqrt{17}} \log\frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$I = \frac{1}{\sqrt{17}} \log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$I = \frac{1}{\sqrt{17}} \log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$I = \frac{1}{\sqrt{17}} \log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$I = \frac{1}{\sqrt{17}} \log\left[\frac{(5+\sqrt{17})(5-\sqrt{17})}{25-17}\right]$$

$$I = \frac{1}{\sqrt{17}} \log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$I = \frac{1}{\sqrt{17}} \log\left[\frac{42+10\sqrt{17}}{8}\right]$$

$$I = \frac{1}{\sqrt{17}} \log\left[\frac{21+5\sqrt{17}}{4}\right]$$

7. $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

उत्तर- $I = \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+4}$

$$I = \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

माना $x+1 = t$

$dx = dt$

जब $x = -1$, $t = 0$ और जब $x = 1$, $t = 2$.

$$I = \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

$$I = \int_0^2 \frac{dt}{(t)^2 + (2)^2}$$

$$I = \left[\frac{1}{2} \tan^{-1} \frac{1}{2}\right]_0^2$$

$$I = \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0$$

$$I = \frac{1}{2}\left(\frac{\pi}{4}\right] = \frac{\pi}{8}$$

8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$

उत्तर- माना $2x = t$

$2dx = dt$

जब $x = 1, t = 2$, और जब $x = 2, t = 4$.

$$I = \int_2^1 \left(\frac{1}{x} - \frac{1}{(t)^2}\right) e^t\, dx$$

$$I = \frac{1}{2} \int_2^1 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t\, dt$$

$$I = \int_2^1 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt$$

यदि $1/t = f(t)$ तो $f'(t) = -1/t^2$.

$$I = \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t \, dt$$

$$I = \int_2^4 e^t \left[f(t) + f'(t)\right] dt$$

$$I = \left[e^t f(t)\right]_2^4$$

$$I = \left[e^t \cdot \frac{2}{t}\right]_2^4$$

$$I = \left[\frac{e^t}{t}\right]_2^4$$

$$I = \frac{e^4}{4} - \frac{e^2}{4}$$

$$I = \frac{e^2(e^2-2)}{4}$$

9. समाकलन $\int_{1/3}^{1} \frac{(x - x^3)^{1/3}}{x^4}$ का मान है :-

(A) 6.
(B) 0.
(C) 3.
(D) 4.

उत्तर-

$$I = \int_{1/3}^{1} \frac{(x - x^3)^{1/3}}{x^4}$$

माना $x = \sin\theta$

$dx = \cos\theta \, d\theta$

जब $x = 1/3$, $\theta = \sin^{-1}(1/3)$ और

जब $x = 1$, $\theta = \pi/2$.

$$I = \int_{\sin^{-1}(1/3)}^{\pi/2} \frac{(\sin\theta - \sin^3\theta)^{1/3}}{\sin^4\theta} \cos\theta \, d\theta$$

$$I = \int_{\sin^{-1}(1/3)}^{\pi/2} \frac{(\sin\theta)^{1/3}(1 - \sin^2\theta)^{1/3}}{\sin^4\theta} \cos\theta \, d\theta$$

$$I = \int_{\sin^{-1}(1/3)}^{\pi/2} \frac{(\sin\theta)^{1/3}(\cos\theta)^{2/3}}{\sin^2\theta \sin^2\theta} \cos\theta \, d\theta$$

$$I = \int_{\sin^{-1}(1/3)}^{\pi/2} \frac{(\cos\theta)^{5/3}}{(\sin\theta)^{5/3}} \operatorname{cosec}^2\theta \, d\theta$$

$$I = \int_{\sin^{-1}(1/3)}^{\pi/2} (\cot\theta)^{5/3} \operatorname{cosec}^2\theta \, d\theta$$

माना $\cot\theta = t$

$-\operatorname{cosec}^2\theta \, d\theta = dt$

जब $\theta = \sin^{-1}(1/3)$, $t = 2\sqrt{2}$, और

जब $\theta = \pi/2$, $t = 0$.

$$I = \int_{2\sqrt{2}}^{0} (t)^{5/3} \, dt$$

$$I = -\left[\frac{3}{8}(t)^{8/3}\right]_{2\sqrt{2}}^{0}$$

$$I = -\frac{3}{8}\left[(-2\sqrt{2})^{8/3}\right]$$

$$I = \frac{3}{8}\left[(\sqrt{8})^{8/3}\right]$$

$$I = \frac{3}{8}\left[(8)^{4/3}\right]$$

$$I = \frac{3}{8} \times 16$$

$I = 3 \times 2 = 6$

अतः विकल्प (A) सही है।

10. यदि $f(x) = \int_0^x t \sin t \, dt$, तब $f'(x)$ है:

(A) $\cos x + x \sin x$
(B) $x \sin x$
(C) $x \cos x$
(D) $\sin x + x \cos x$

उत्तर- $f(x) = \int_0^x t \sin t \, dt$

$$f(x) = t \int_0^x \sin t \, dt - \int_0^x \left\{ \left(\frac{d}{dt} t \right) \int \sin t \, dt \right\} dt$$

$$f(x) = \left[t(-\cos t) \right]_0^x - \int_0^x (-\cos t) \, dt$$

$$f(x) = \left[-t \cos t + \sin t \right]_0^x$$

$$f(x) = -x \cos x + \sin x$$

$$f'(x) = -\left[\{ x(-\sin x) \} + \cos x \right] + \cos x$$

$$f'(x) = x \sin x - \cos x + \cos x$$

$f'(x) = x \sin x$

अतः विकल्प (B) सही है।

प्रश्नावली 7.8

1. $\int_{-1}^{1} (x+1)\,dx$

उत्तर- $I = \int_{-1}^{1} (x+1)\,dx$

$$I = \left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$I = \left(\frac{1}{2} + 1\right) - \left(\frac{1}{2} - 1\right)$$

$$I = \frac{1}{2} + 1 - \frac{1}{2} + 1$$

$$I = 2.$$

2. $\int_{2}^{3} \frac{1}{x}\,dx$

उत्तर- $I = \int_{2}^{3} \frac{1}{x}\,dx$

$$I = [\log x]_{2}^{3}$$

$I = \log|3| - \log|2|$

$I = \log\left|\frac{3}{2}\right|$

3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\,dx$

उत्तर- $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9)\,dx$

$I = \left[4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)\right]_1^2$

$I = \left[x^4 - \frac{5x^3}{3} + 3x^2 + 9x\right]_1^2$

$I = \left\{2^4 - \frac{5(2)^3}{3} + 3(2)^2 + 9(2)\right\} -$

$\left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$

$I = \left\{16 - \frac{40}{3} + 12 + 18\right\} -$

$\left\{1 - \frac{5}{3} + 3 + 9\right\}$

$I = 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$

$$I = 33 - \frac{35}{3}$$

$$I = \frac{64}{3}$$

4. $\int_0^{\pi/4} \sin 2x \, dx$

उत्तर- $I = \int_0^{\pi/4} \sin 2x \, dx$

$$I = \left[\frac{-\cos 2x}{2}\right]_0^{\pi/4}$$

$$I = \frac{-1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$$

$$I = \frac{-1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$I = -\frac{1}{2}(0-1) = \frac{1}{2}$$

5. $\int_0^{\pi/4} \cos 2x \, dx$

उत्तर- $I = \int_0^{\pi/4} \cos 2x \, dx$

$$I = \left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$I = \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$I = \frac{1}{2}\left[\sin \pi - \sin 0\right]$$

$$I = \frac{1}{2}(0-0)$$

$$I = 0$$

6. $\int_4^5 e^x dx$

उत्तर- $I = \int_4^5 e^x dx$

$$I = \left[e^x\right]_4^5$$

$$I = e^5 - e^4$$

$$I = e^4(e-1)$$

उत्तर- $I = \int_0^{\pi/4} \tan x \, dx$

$I = \left[-\log |\cos x| \right]_0^{\pi/4}$

$I = -\log \left| \cos \frac{\pi}{4} \right| + \log |\cos 0|$

$I = -\log \left| \frac{1}{\sqrt{2}} \right| + \log |1|$

$I = -\log (2)^{-1/2}$

$I = \frac{1}{2} \log 2.$

8. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx$

उत्तर- $I = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx$

$I = \left[\log |\operatorname{cosec} x - \cot x| \right]_{\pi/6}^{\pi/4}$

$I = \log \left| \operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4} \right| - \log$

$$\left| \operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6} \right|$$

$$I = \log |\sqrt{2}-1| - \log |2-\sqrt{3}|$$

$$I = \log \left(\frac{\sqrt{2}-1}{2-\sqrt{3}} \right)$$

9. $$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

उत्तर- $$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$I = \left[\sin^{-1} x \right]_0^1$$

$$I = \sin^{-1}(1) - \sin^{-1}(0)$$

$$I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{2}$$

10. $$\int_0^1 \frac{dx}{1+x^2}$$

उत्तर- $$I = \left[\tan^{-1} x \right]_0^1$$

$$I = \tan^{-1}(1) - \tan^{-1}(0)$$

$$I = \frac{\pi}{4}$$

11. $\int_0^1 \frac{dx}{x^2-1}$

उत्तर- $I = \left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_0^1$

$$I = \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{3-1}{3+1}\right|\right]$$

$$I = \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$I = \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$

$$I = \frac{1}{2}\left[\log\frac{3}{2}\right]$$

12. $\int_0^{\pi/2} \cos^2 x\, dx$

उत्तर- $I = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx$

$$I = \left[\frac{x}{2} + \sin 2x/4\right]_0^{\pi/2}$$

$$I = \left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right]_0^{\pi/2}$$

$$I = \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin \pi}{2}\right)\right]$$

$$I = \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$$

$$I = \frac{\pi}{4}.$$

3. $$\int_2^3 \frac{x\,dx}{x^2+1}$$

2. $$I = \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\,dx$$

$$I = \left[\frac{1}{2}\log\left(1+x^2\right)\right]_2^3$$

$$I = \frac{1}{2}\left[\log\left(1+(3)^2\right] - \left[\log(1+(2)\right.\right.$$

$$I = \frac{1}{2}\left[\log(10) - \log(5)\right]$$

$$I = \frac{1}{2}\left[\log \frac{10}{5}\right] = \frac{1}{2}\log 2.$$

14. $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$

उत्तर-

$$I = \int_0^1 \frac{2x+3}{5x^2+1}\,dx$$

$$I = \frac{1}{5}\int_0^1 \frac{5(2x+3)}{5x^2+1}\,dx$$

$$I = \frac{1}{5}\int_0^1 \frac{10x+15}{5x^2+1}\,dx$$

$$I = \frac{1}{5}\int \frac{10x}{5x^2+1}\,dx + 3\int \frac{1}{5x^2+1}\,dx$$

$$I = \frac{1}{5}\int \frac{10x}{5x^2+1}\,dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}\,dx$$

$$I = \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$I = \frac{1}{5}\log(5x^2+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5}x)$$

$$I = \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$I = \frac{1}{5}\log(6) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})$$

15. $\int_0^1 x e^{x^2}\,dx$

उत्तर- $I = \int_0^1 x e^{x^2}\,dx$

$x^2 = t$

$2x\,dx = dt$

यदि $x \to 0, t \to 0$ और

यदि $x \to 1, t \to 1$

$$I = \frac{1}{2}\int_0^1 e^t\,dt$$

$$I = \frac{1}{2}\left[e^t\right]_0^1$$

$$I = \frac{1}{2}e - \frac{1}{2}e^0$$

$$I = \frac{1}{2}(e-1)$$

17. $\int_0^{\pi/4} (2\sec^2 x + x^3 + 2)\,dx$

उत्तर- $I = \left[2\tan x + \frac{x^4}{4} + 2x\right]_0^{\pi/4}$

$$I = \left[\left\{2\tan \pi/4 + 1/4\,(\pi/4)^4 + 2(\pi/4)\right\} - \left\{2\tan 0 + 0 + 0\right\}\right]$$

$$I = 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$I = 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

18. $\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{\pi}{2}\right)dx$

उत्तर- $I = -\int_0^{\pi} \left(\cos^2\frac{x}{2} - \sin^2\frac{\pi}{2}\right)dx$

$$I = -\int_0^{\pi} \cos x\,dx$$

$I = -[\sin x]_0^{\pi}$

$I = -[\sin \pi - \sin 0]$

$I = -[0 - 0]$

$I = 0.$

19. $\int_0^2 \frac{6x+3}{x^2+4}\,dx$

3+2-

$I = 3\int_0^2 \frac{2x+1}{x^2+4}\,dx$

$I = 3\int_0^2 \frac{2x}{x^2+4}\,dx + 3\int_0^2 \frac{1}{x^2+4}$

$I = \left[3\log(x^2+4) + \frac{3}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$I = \{3\log(2^2+4) + 3/2\tan^{-1}(2/2)\}$
$- \{3\log(0+4) + 3/2\tan^{-1}(0/2)\}$

$I = 3\log 8 + 3/2\tan^{-1}1 - 3\log 4$
$- 3/2\tan^{-1}0$

$I = 3\log 8 + 3/2\tan^{-1}(\pi/4) -$
$3\log 4 - 0$

$$I = 3 \log\left(\frac{8}{4}\right) + \frac{3\pi}{4}$$

$$I = 3 \log 2 + 3\pi/4$$

21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ बराबर है :

(A) $\pi/3$.
(B) $2\pi/3$.
(C) $\pi/6$.
(D) $\pi/12$.

उत्तर- $I = \int_1^{\sqrt{3}} \frac{dx}{1+x^2}$

$$I = \left[\tan^{-1} x\right]_1^{\sqrt{3}}$$

$$I = \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$I = \frac{\pi}{3} - \frac{\pi}{4}$$

$$I = \frac{\pi}{12}.$$

22. $\int_0^{2/3} \frac{dx}{4+9x^2}$ बराबर है :

(A) $\pi/6$

(B) $\pi/12$

(C) $\pi/24$

(D) $\pi/4$

उत्तर- $I = \int_0^{2/3} \frac{dx}{4+9x^2} = \int_0^{2/3} \frac{dx}{(2)^2+(3x)^2}$

माना $3x = t$

$3dx = dt$

$= \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dx}{(2)^2+(t)^2}$

$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right] = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$

$I = \left[\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{2/3}$

$I = \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$

$I = \frac{1}{6} \times \frac{\pi}{4} = \frac{\pi}{24}$.

अतः विकल्प (c) सही है।

निश्चित समाकलन

(1) $P_0 : \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(2) $P_1 : \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$,

विशिष्टतया $\int_b^a f(x)\,dx = 0$.

(3) $P_2 : \int_a^b f(x)\,dx = \int_a^c f(x)\,dx$,
a, b, c वास्तविक संख्याएँ हैं।

(4) $P_3 : \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(5) $P_4 : \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(6) $P_5 : \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(7) $P_6 : \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

यदि $f(2a-x) = f(x) = 0$

या $f(2a-x) = -f(x)$

(8) P_7 : (i) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, यदि f एक सम फलन है या $f(-x) = f(x)$

(ii) $\int_{-a}^{a} f(x)\,dx = 0$, यदि f एक विषम फलन है या $f(-x) = -f(x)$.

प्रश्नावली - 7.10

1. $\int_{0}^{\pi/2} \cos^2 x\,dx$

उत्तर- $I = \int_{0}^{\pi/2} \cos^2 x\,dx$ — समी. 1

$I = \int_{0}^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx$

$\left[\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx\right]$

$I = \int_{0}^{\pi/2} \sin^2 x\,dx$ — समी. 2

समी. 1 और 2 को जोड़ने पर

$2I = \int_{0}^{\pi/2} (\sin^2 x + \cos^2 x)\,dx$

$$2I = \int_0^{\pi/2} 1 \, dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}.$$

2. $$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$$

उत्तर-

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx \quad \text{— समी.1}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \, dx$$

$$\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \, dx \quad \text{— समी.2}$$

समी॰ 1 और 2 को जोड़ने पर :-

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx$$

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$2I = \int_0^{\pi/2} 1 \, dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

3. $$\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx$$

उत्तर-

$$I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \quad \text{— समी॰}$$

$$I = \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right)+\cos^{3/2}\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\left[\int_a^a f(x)\,dx = \int_0^a f(a-x)\right]$$

$$I = \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x}\,dx$$

समी॰1 और 2 को जोड़ने पर :-

$$2I = \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\,dx$$

$$2I = \int_0^{\pi/2} 1\,dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}.$$

4. $\int_0^{\pi/2} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x} \, dx$

उत्तर-

$$I = \int_0^{\pi/2} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x} \quad \text{— समी. 1}$$

$$I = \int_0^{\pi/2} \frac{\cos^5\left(\frac{\pi}{2} - x\right) dx}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} \, dx$$

$$\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \int_0^{\pi/2} \frac{\sin^5 x \, dx}{\cos^5 x + \sin^5 x} \, dx \quad \text{— समी. 2}$$

समी. 1 और 2 को जोड़ने पर :-

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\cos^5 x + \sin^5 x} \, dx$$

$$2I = \int_0^{\pi/2} 1 \, dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

5. $\int_{-5}^{5} |x+2|\, dx$

उत्तर- $I = \int_{-5}^{5} |x+2|\, dx$

यहाँ $(x+2) < 0$ यदि $x < -2$ और

$(x+2) > 0$ यदि $x > -2$.

$$I = \int_{-5}^{-2} -(x+2)\, dx + \int_{-2}^{5} (x+2)\, dx$$

$$\left[\int_a^b f(x) = \int_a^e f(x) + \int_e^b f(x)\right]$$

$$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$I = -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(2)^2}{2} + 2(2)\right]$$

$$I = -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$I = -2 + 4 + \frac{25}{2} + 10 - 2 + 4$$

$$I = 29.$$

$$\int_2^8 |x-5|\, dx$$

$$I = \int_2^8 |x-5|\, dx$$

यहाँ $(x-5) < 0$ यदि $x < 5$ और

$(x-5) > 0$ यदि $x > 5$.

$$I = \int_2^5 -(x-5)\, dx + \int_2^8 (x-5)\, dx$$

$$\left[\int_a^b f(x) = \int_a^e f(x) + \int_e^b f(x)\right]$$

$$I = -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$I = -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$I = 9.$$

7. $\int_0^1 x(1-x)^n \, dx$

उत्तर- $I = \int_0^1 x(1-x)^n \, dx$

$$I = \int_0^1 (1-x)\left(1-(1-x)\right)^n \, dx$$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^1 (1-x)(x)^n \, dx$$

$$I = \int_0^1 (x^n - x^{n+1}) \, dx$$

$$I = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$I = \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$I = \frac{(n+2)-(n+1)}{(n+1)(n+2)}$$

$$I = \frac{1}{(n+1)(n+2)}$$

8. $\int_0^{\pi/4} \log(1+\tan x)\,dx$

उत्तर- $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ — समी० 1

$$I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$$

$$I = \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$I = \int_0^{\pi/4} \log \frac{2}{(1+\tan x)}\, dx$$

$$I = \int_0^{\pi/4} \log 2\, dx - \int_0^{\pi/4} \log (1+\tan x)\, dx$$

$$I = \int_0^{\pi/4} \log 2\, dx - I \quad \text{(समी० 1 से)}$$

$$2I = [x \log 2]_0^{\pi/4}$$

$$2I = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2.$$

9. $\int_0^2 x\sqrt{2-x}\,dx$

उत्तर- $I = \int_0^2 x\sqrt{2-x}\,dx$

$I = \int_0^2 (2-x)\sqrt{x}\,dx$

$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^2 \left\{2x^{1/2} - x^{3/2}\right\} dx$

$I = \left[2\left(\frac{x^{3/2}}{3/2}\right) - \left(\frac{x^{5/2}}{5/2}\right)\right]_0^1$

$I = \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$

$I = \frac{4}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2}$

$I = \frac{4 \times 2\sqrt{2}}{3} - \frac{2}{5} \times 4\sqrt{2}$

$$I = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$I = \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$I = \frac{16\sqrt{2}}{15}$$

10. $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$

उत्तर- $I = \int_0^{\pi/2} \{(2 \log \sin x - \log (2 \sin x \cos x)\}\, dx$

$$I = \int_0^{\pi/2} \{(2 \log \sin x - \log \sin x - \log \cos x - \log 2)\}\, dx$$

$$I = \int_0^{\pi/2} \{(\log \sin x - \log \cos x - \log 2)\}\, dx \quad \ldots(1)$$

$$\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \int_0^{\pi/2} \{(\log \cos x - \log \sin x - \log 2)\}\, dx \quad \ldots(2)$$

समी० 1 व 2 को जोड़ने पर :-

$$2I = \int_0^{\pi/2} (-\log 2 - \log 2)\, dx$$

$$2I = -2 \log 2 \int_0^{\pi/2} 1\, dx$$

$$I = -\log 2 \left[\frac{\pi}{2}\right]$$

$$I = \frac{\pi}{2}(-\log 2)$$

$$I = -\frac{\pi}{2} \log 2.$$

11. $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$

उत्तर-

$$I = \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$$

$\sin^2(-x) = (\sin(-x))^2 = \sin^2 x$

$\sin^2 x$ एक सम फलन है।

$f(x)$ एक सम फलन है तो,

$$\left[\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right]$$

$$I = 2\int_{0}^{\pi/2} \sin^2 x\,dx$$

$$I = 2\int_{0}^{\pi/2} \frac{1-\cos 2x}{2}\,dx$$

$$I = \int_{0}^{\pi/2} (1-\cos 2x)\,dx$$

$$I = \left[x - \frac{\sin 2x}{2}\right]_{0}^{\pi/2}$$

$$I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{2}.$$

12. $\int_{0}^{\pi} \frac{x\,dx}{1+\sin x}$

उत्तर-

$$I = \int_{0}^{\pi} \frac{x\,dx}{1+\sin x} \quad \ldots (1)$$

$$I = \int_{0}^{\pi} \frac{(\pi-x)\,dx}{1+\sin(\pi-x)}$$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\sin x} \quad \ldots (2)$$

समी. 1 और 2 को जोड़ने पर :-

$$2I = \int_0^{\pi} \frac{\pi\,dx}{1+\sin x}$$

$$2I = \pi \int_0^{\pi} \frac{(1-\sin x)\,dx}{(1+\sin x)(1-\sin x)}$$

$$2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\}\,dx$$

$$2I = \pi \left[\tan x \sec x\right]_0^{\pi}$$

$$2I = \pi [2]$$

$$I = \pi.$$

13. $$\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$$

उत्तर- $I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx \quad \ldots (1)$

$\sin^7(-x) = [(\sin(-x))]^7 = (-\sin x)^2 =$

$-\sin^7 x$, $\sin^7 x$ एक विषम फलन है।

$f(x)$ एक विषम फलन हो तो,

$$\int_{-a}^{a} f(x)\, dx = 0$$

$$I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx = 0.$$

14. $\int_0^{2\pi} \cos^5 x \, dx$

उत्तर- $I = \int_0^{2\pi} \cos^5 x \, dx \quad \ldots (1)$

$$I = \int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx \\ 0 \end{cases}$$

यदि $f(2a-x) = f(x)$

यदि $f(2a-x) = -f(x)$

$\cos^5(2\pi - x) = \cos^5 x$,

$\therefore I = 2\int_0^{\pi} \cos^5 x \, dx$

$\cos^5(\pi - x) = -\cos^5 x$,

$\therefore I = 2(0) = 0$

15. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

उत्तर-

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \quad \ldots (1)$$

$$I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x) - \cos(\pi/2 - x)}{1 + \sin(\pi/2 - x)\cos(\pi/2 - x)} dx$$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx \quad \ldots (2)$$

समी. 1 और 2 को जोड़ने पर

$$2I = \int_0^{\pi/2} \frac{0}{1 + \sin x \cos x} dx$$

$2I = 0$

$I = 0.$

16. $\int_0^{\pi} \log(1+\cos x)\,dx$

उत्तर. $I = \int_0^{\pi} \log(1+\cos x)\,dx \quad \ldots(1)$

$I = \int_0^{\pi} \log(1+\cos(\pi - x))\,dx$

$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^{\pi} \log(1-\cos x)\,dx \quad \ldots(2)$

समी. 1 और 2 को जोड़ने पर

$2I = \int_0^{\pi} \{\log(1+\cos x) + \log(1-\cos x)\}\,dx$

$2I = \int_0^{\pi} \log(1-\cos^2 x)\,dx$

$2I = \int_0^{\pi} \log(\sin^2 x)\,dx$

$$2I = 2\int_0^{\pi} \log(\sin x)\, dx$$

$$I = \int_0^{\pi} \log(\sin x)\, dx \quad \ldots(3)$$

$$\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^{a} f(x)\, dx \\ 0 \end{cases}$$

यदि $f(2a-x) = f(x)$

यदि $f(2a-x) = -f(x)$

$\sin(\pi - x) = \sin x$

$$I = 2\int_0^{\pi/2} \log \sin x\, dx \quad \ldots(4)$$

$$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$I = 2\int_0^{\pi/2} \log \cos x\, dx \quad \ldots(5)$$

समी. 4 और 5 को जोड़ने पर

$$2I = 2\int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$I = \int_0^{\pi/2} (\log \sin x + \log \cos x + \log 2 - \log 2)\, dx$$

$$I = \int_0^{\pi/2} (\log 2 \sin x \cos x - \log 2)\, dx$$

$$I = \int_0^{\pi/2} \log 2 \sin x \cos x \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$I = \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 [x]_0^{\pi/2}$$

$$I = \int_0^{\pi/2} \log \sin 2x \, dx - \frac{\pi}{2} \log 2$$

माना $2x = t$

$2\, dx = dt$

जब $x = 0, t = 0$ और

जब $x = \pi/2, t = \pi$

$$I = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$I = \frac{1}{2} I - \frac{1}{2} \log 2$$

$$\frac{1}{2} I = \frac{-\pi}{2} \log 2$$

$$I = -\pi \log 2.$$

17. $$\int_0^a \frac{\sqrt{x}}{\sqrt{x} - \sqrt{a-x}} \, dx$$

उत्तर- $$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} \, dx \quad \ldots (1)$$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \, dx \quad \ldots (2)$$

$$\left[\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right]$$

समी. 1 और 2 को जोड़ने पर

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} \, dx$$

$$2I = \int_0^a 1 \, dx$$

$$2I = \left[x \right]_0^a$$

$2I = a$

$I = \frac{a}{2}$.

18. $\int_0^4 |x-1|\,dx$

उत्तर- $I = \int_0^4 |x-1|\,dx$

$(x-1) < 0, \; x < 1, \; (x-1) > 0, \; x > 1$

$I = \int_0^1 -(x-1)\,dx + \int_1^4 (x-1)\,dx$

$\left[\int_a^b f(x) = \int_a^e f(x) + \int_e^b f(x)\right]$

$I = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$

$I = 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1$

$I = 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1$

$I = 5.$

19. दर्शाइए कि $\int_0^a f(x)\,g(x)\,dx = 2\int_0^a f(x)\,dx$, यदि f और g को $f(x) = f(a-x)$ एवं $g(x) + g(a-x) = 4$ के रूप में परिभाषित किया गया है।

उत्तर- $I = \int_0^a f(x)\,g(x)\,dx \quad \ldots (1)$

$I = \int_0^a f(a-x)\,g(a-x)\,dx$

$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^a f(x)\,g(a-x)\,dx \quad \ldots (2)$

समी. 1 और 2 को जोड़ने पर :-

$2I = \int_0^a \{f(x)\,g(x) + f(x)\,g(a-x)\}\,dx$

$2I = \int_0^a \{f(x)\{g(x) + g(a-x)\}\}\,dx$

$2I = \int_0^a f(x) \times 4\,dx$

[दिया है : $g(x) + g(a-x) = 4$]

$$I = 2\int_0^a f(x)\,dx$$

20. $\int_{-\pi/2}^{\pi/2} (x^3\,dx + x\cos x + \tan^5 x + 1)\,dx$ का मान है :

(A) 0
(B) 2
(C) π
(D) 1

उत्तर-

$$I = \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\,dx$$

$$I = \int_{-\pi/2}^{\pi/2} x^3\,dx + \int_{-\pi/2}^{\pi/2} x\cos x\,dx +$$

$$\int_{-\pi/2}^{\pi/2} \tan^5 x\,dx + \int_{-\pi/2}^{\pi/2} 1\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx \\ 0 \end{cases}$$

यदि $f(x)$ एक सम फलन है

यदि $f(x)$ एक विषम फलन है

$$I = 0 + 0 + 0 + 2\int_0^{\pi/2} 1\,dx$$

$$I = 2\left[x\right]_0^{\pi/2}$$

$$I = \frac{2\pi}{2}$$

$$I = \pi$$

अतः विकल्प (c) सही है।

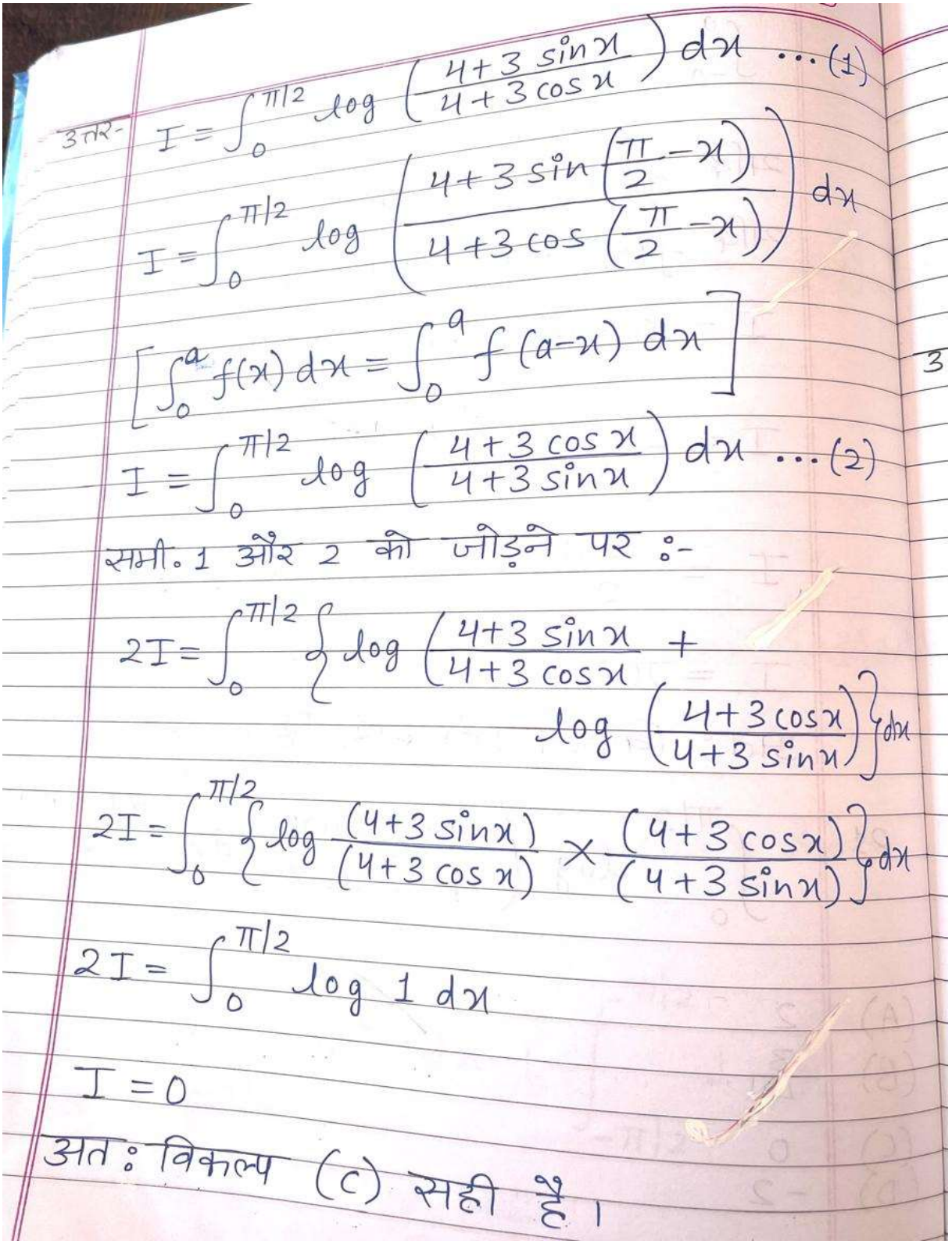
21. $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ का मान है:

) $\frac{2}{3}$

) $\frac{3}{4}$

) 0

) -2



उत्तर-

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx \quad \ldots(1)$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right) dx$$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx \quad \ldots(2)$$

समी. 1 और 2 को जोड़ने पर :-

$$2I = \int_0^{\pi/2} \left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right\} dx$$

$$2I = \int_0^{\pi/2} \left\{\log \frac{(4+3\sin x)}{(4+3\cos x)} \times \frac{(4+3\cos x)}{(4+3\sin x)}\right\} dx$$

$$2I = \int_0^{\pi/2} \log 1 \, dx$$

$$I = 0$$

अतः विकल्प (c) सही है।